ON GEOMETRICALLY TRANSITIVE HOPF ALGEBROIDS

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Abstract. This paper contributes in the characterization of a certain class of commutative Hopf algebroids. It is shown that a given commutative flat Hopf algebroid with a non trivial base ring and a non empty characters groupoid, is geometrically transitive if and only if any base change morphism is a weak equivalence (in particular, any extension of the base ring is Landweber exact), if and only if any trivial bundle is a principal bi-bundle, and if and only if any two objects are fpqc locally isomorphic. As a consequence, any two isotropy Hopf algebras of geometrically transitive Hopf algebroid are weakly equivalent. Furthermore, the characters groupoid is transitive if and only if any two isotropy Hopf algebras are conjugated. Several others characterizations of these Hopf algebroids, in relation with transitive abstract groupoids, are also displayed.

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1. Introduction

1.1. **Motivation and overview.** A commutative Hopf algebroid can be taught as an affine groupoid scheme, that is, groupoid scheme [10, Définition page 299] in which the schemes defining objects and morphisms are affine schemes. In other words, this is a representable presheaf of groupoids in the category of affine schemes, or a prestack of groupoids whose "stackification" leads to a stack in the *fpqc* (fidèlment plate quasi-compacte) topology. For instance, an action of an affine group scheme on affine scheme leads to

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an affine groupoid scheme which gives arise, by passage to the coordinate rings, to a commutative Hopf algebroid, known as *split Hopf algebroid* ([25, Appendix A.1], see also [18]).

Hopf algebroids in relation with groupoids are fundamental objects in both algebraic topology and algebraic geometry. They patently appear in the study of stable homotopy theory [25, 16, 23, 24] (see also the references therein), and prove to be very useful in studying quotient of preschemes, prestacks of groupoids over affine schemes as well as (commutative) Tannakian categories [13, 9, 4, 1, 19].

As in the case of affine group schemes [10], several constructions and results on abstract groupoids have a certain geometric meaning in presheaves of groupoids and then a possible algebraic interpretation at the level of Hopf algebroids. In this way, Hopf algebroids are better understood when looking to classical results in abstract groupoids, or by mimicking well known results on classical geometric groupoids, e.g. topological or Lie groupoids. The main motivation of this paper fits into these lines of research. Specifically, we study a class of commutative Hopf algebroids called *geometrically transitive* (see below), by means of transitive groupoids and theirs properties, obtaining in this way several new characterizations of these Hopf algebroids. Beside, much of the properties of transitive groupoids hereby developed and employed in the study of Hopf algebroids, can be also seen as a contribution to the theory of abstract groupoids.

The notion of transitivity varies depending on the context. In abstract groupoids, a (small) groupoid is said to be *transitive* when the cartesian product of the source and the target is a surjective map. A Lie groupoid is called *locally trivial* (or a *transitive Lie groupoid*), when this map is a surjective submersion [20, 8]. For groupoids schemes, the meaning of the abstract notion of transitivity was introduced by Deligne in [9, page 114]. Precisely, a groupoid scheme is *transitive*, in the fpqc topology sense, if the morphism performed by the fibred product of the source and the target is a cover in this topology. In [4, Définition page 5850], Bruguières introduced a class of non commutative Hopf algebroids referred to as *geometrically transitive*, where he showed that in the commutative case (the case which we are interested in) these are Hopf algebroids whose associated affine groupoid schemes are transitive in the fpqc sense. It is also implicitly shown in [4] that a commutative Hopf algebroid is geometrically transitive if and only if the unit map (i.e. the tensor product of the source and the target) is a faithfully flat extension. This in fact can be taught as a proper definition of geometrically transitive commutative (GT for short) Hopf algebroid. Nevertheless, we will use here the original definition of [4] (see Definition 4.2 below) and show using elementary methods that the faithfully flatness of the unit characterizes in fact GT Hopf algebroids.

Transitive groupoids are also characterized by the fact that any two objects are isomorphic, equivalently a groupoid with only one connected component, a *connected groupoid* in the terminology of [15]. From the geometric point of view, that is for presheaves of groupoids, this means that any two objects are locally isomorphic in the fpqc topology, see [9, Proposition 3.3]. At the level of Hopf algebroids, this property can be directly expressed in terms of faithfully flat extensions (see Definition 3.5 below), which in turns characterizes GT Hopf algebroids, as we will see, in the sequel, by using elementary (algebraic) arguments.

It is then from their own definition that GT Hopf algebroids can be viewed as natural algebro-geometric substitute notion of transitive abstract groupoids. However, still several characterizations of transitive groupoids which, up to our knowledge, are not known for GT Hopf algebroids. In the following, we describe the two most notably of these characterizations.

A perhaps well known result (see Proposition 2.15, for details) says that a groupoid $\mathscr{G}$: $G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0$ is transitive if and only if, for any map $\varsigma : X \to G_0$, the induced morphism of groupoids $\mathscr{G}^{\varsigma} \to \mathscr{G}$ is a weak equivalence (i.e. essentially surjective and fully faithful functor), where $\mathscr{G}^{\varsigma}$ is the *induced groupoid* whose set of objects is X and its set of arrows is the fibred product $X\,{}_{\varsigma}\!\times_{t} G_1\,{}_{s}\!\times_{\varsigma} X$, that is, $\mathscr{G}^{\varsigma}$ is the pull-back groupoid of $\mathscr{G}$ along ς. A more interesting, and perhaps not yet known, is the characterization of the transitivity by means of *principal groupoids-bisets*; for the precise definition of this notion see Definitions 2.6, 2.7 and 2.8. This notion is in fact an abstract formulation of the notion of *principal bi-bundles* in topological and Lie groupoids context [21, 17], or that of *bi-torsor* in sheaves theory [10, 14], which is of course based on the natural generalization of the notion of group bisets [7] to the context of groupoids. The aforementioned characterization is expressed as follows: a groupoid $\mathscr{G}$ is transitive if and only if, for any map $\varsigma : X \to G_0$, the pull-back groupoid-biset $G_1\,{}_{s}\!\times_{\varsigma} X$ is a principal $(\mathscr{G}, \mathscr{G}^{\varsigma})$-biset.

The main aim of this paper is to investigate GT Hopf algebroids by means of transitive abstract groupoids. Our aim is in part to see how the previous characterizations of transitive groupoids, by means of weak equivalences and principal groupoids-bisets, can be transferred to the commutative Hopf algebroids framework.

1.2. **Description of the main results.** Let $\Bbbk$ be a ground base field. An algebra stands for commutative $\Bbbk$-algebra, and unadorned tensor product denote the tensor product between $\Bbbk$-vector spaces.

Our main result is summarised in the following theorem which includes Theorem 4.6 below:

Theorem **A**. *Let $(A, \mathcal{H})$ be a commutative flat Hopf algebroid over $\Bbbk$ and denote by $\mathscr{H}$ its associated presheaf of groupoids. Assume that $A \neq 0$ and $\mathscr{H}_0(\Bbbk) = \mathrm{Alg}_{\Bbbk}(A, \Bbbk) \neq \emptyset$. Then the following are equivalent:*

(i) $\mathsf{s} \otimes \mathsf{t} : A \otimes A \to \mathcal{H}$ *is a faithfully flat extension;*
(ii) Any two objects of $\mathscr{H}$ are locally isomorphic (see Definition 3.5);
(iii) For any extension $\phi : A \to B$, the extension $\alpha : A \to \mathcal{H}_{\mathsf{t}} \otimes_{A\ \phi} B$, $a \mapsto \mathsf{s}(a) \otimes_A 1_B$ is faithfully flat;
(iv) $(A, \mathcal{H})$ is geometrically transitive (see Definition 4.2);
(v) For any extension $\phi : A \to B$, the canonical morphism $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{H}_\phi)$ of Hopf algebroids is a weak equivalence, that is, the induced functor $\phi_* : \mathsf{Comod}_{\mathcal{H}} \to \mathsf{Comod}_{\mathcal{H}_\phi}$ is an equivalence of symmetric monoidal categories of comodules, where $\mathcal{H}_\phi = B \otimes_A \mathcal{H} \otimes_A B$;
(vi) The trivial principal left $(\mathcal{H}, \mathcal{H}_\phi)$-bundle $\mathcal{H} \otimes_A B$ is a principal bi-bundle (see subsection 3.4).

The fact that transitive groupoids are characterized by the conjugacy of their isotropy groups, and the analogue of this characterization in the Hopf algebroids context, also attract our attention. More precisely, given a commutative Hopf algebroid $(A, \mathcal{H})$ with a non empty characters groupoid, that is, $\mathscr{H}_0(\Bbbk) \neq \emptyset$. Then for any object $x : A \to \Bbbk$ of this (fibre) groupoid, there is a presheaf which to each commutative algebra C attached the isotropy group of the object $x^*(1_C) \in \mathscr{H}_0(C)$, where $1_C : \Bbbk \to C$ is the unit of C. It turns out that this is an affine group scheme represented by the Hopf algebra $(\Bbbk_x, \mathcal{H}_x)$ which is the base change of $(A, \mathcal{H})$ by the algebra map x (here $\Bbbk_x$ denotes $\Bbbk$ viewed as an A-algebra via x, and $\mathcal{H}_x = \Bbbk_x \otimes_A \mathcal{H} \otimes_A \Bbbk_x$). The pair $(\Bbbk_x, \mathcal{H}_x)$ is refereed to as the *isotropy Hopf algebra* of $(A, \mathcal{H})$ at (the point) x.

Recall from [16] that two flat Hopf algebroids $(A, \mathcal{H})$ and $(B, \mathcal{K})$ are weakly equivalent if there is a diagram of weak equivalences

$$\begin{tikzcd} & (C, \mathcal{J}) & \\ (A, \mathcal{H}) \arrow[ur] & & (B, \mathcal{K}), \arrow[ul] \end{tikzcd}$$

see subsection 3.4 for more details. The fact that two isotropy groups of transitive groupoids are isomorphic, is translated to the fact that two isotropy Hopf algebras of GT Hopf algebroid are weakly equivalent; of course, Hopf algebras are considered here as Hopf algebroids with source equal target.

This result is part of the subsequent corollary of Theorem **A** which contains both Proposition 5.3 and Corollary 5.14:

Corollary **A**. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid as in Theorem* **A** *. Assume that $(A, \mathcal{H})$ is geometrically transitive. Then*

(1) Any two isotropy Hopf algebras are weakly equivalent.
(2) Any dualizable (right) $\mathcal{H}$-comodule is locally free of constant rank. Moreover, any right $\mathcal{H}$-comodule is an inductive limit of dualizable right $\mathcal{H}$-comodules.

The notion of the conjugation relation between two isotropy Hopf algebras is not automatic. This relation can be formulated by using 2-isomorphisms in the 2-category of flat Hopf algebroids. Precisely, using the notations and the assumptions of Theorem **A**, for a given Hopf algebroid $(A, \mathcal{H})$, two isotropy Hopf algebras $(\Bbbk_x, \mathcal{H}_x)$ and $(\Bbbk_y, \mathcal{H}_y)$, at $x, y \in \mathscr{H}_0(\Bbbk)$, are said to be *conjugated* provided there is an isomorphism $\mathsf{g} : (\Bbbk_x, \mathcal{H}_x) \to (\Bbbk_y, \mathcal{H}_y)$ of Hopf algebras such that the following diagram

$$\begin{tikzcd} (\Bbbk_x, \mathcal{H}_x) \arrow[rr, "\mathsf{g}"] & & (\Bbbk_y, \mathcal{H}_y) \\ & (A, \mathcal{H}) \arrow[ul, "\mathsf{x}"] \arrow[ur, "\mathsf{y}"'] & \end{tikzcd}$$

commutes up to a 2-isomorphism, where x and y are the canonical morphisms attached, respectively, to x and y. The transitivity of the conjugation relation characterizes in fact the transitivity of the characters groupoid. This result is also a corollary of Theorem **A** and stated as Proposition 5.8 below:

Corollary **B**. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid as in Theorem* **A**. *Assume that $(A, \mathcal{H})$ is geometrically transitive. Then, the following are equivalent*

(i) *the characters groupoid $\mathscr{H}(\Bbbk)$ is transitive;*
(ii) *for any two objects x, y in $\mathscr{H}_0(\Bbbk)$, the left $\mathcal{H}$-comodules algebras $\mathcal{H} \otimes_A \Bbbk_x$ and $\mathcal{H} \otimes_A \Bbbk_y$ are isomorphic;*
(iii) *any two isotropy Hopf algebras are conjugated.*

Over an algebraically closed field $\Bbbk$, we observe that the characters groupoids $\mathscr{H}(\Bbbk)$ of any GT Hopf algebroid $(A, \mathcal{H})$ with finitely generated total algebra $\mathcal{H}$, is always transitive. Thus, the equivalent conditions in Corollary **B** are satisfied in this case.

Transitive groupoids are related to principal group-bisets, in the sense that there is a (non canonical) correspondence between these two notions, see subsection 2.6. This in fact is an abstract formulation of the Ehresmann's well known result dealing with the correspondence between transitive Lie groupoids and principal fiber bundles, as was expounded in [8].

The analogue correspondence at the level of Hopf algebroids is not always possible and some technical assumptions are required. The formulation of this result is given as follows.

For any object $x \in \mathscr{H}_0(\Bbbk)$, consider the presheaf of sets which associated to each commutative algebra C the set $\mathscr{P}_x(C) := \mathsf{t}^{-1}(\{x^*(1_C)\})$, where t is the target of the groupoid $\mathscr{H}(C)$. In the terminology of [15] this is the *left star set* of the object $x^*(1_C)$. Denotes by $\alpha_x : A \to P_x := \mathcal{H} \otimes_A \Bbbk_x$ the algebra map which sends $a \mapsto \mathsf{s}(a) \otimes 1$. It turns out that the presheaf of sets $\mathscr{P}_x$ is affine, and up to a natural isomorphism, is represented by the $(\mathcal{H}, \mathcal{H}_x)$-bicomodule algebra P_x. The subsequent is a corollary of Theorem **A**, and formulates the desired result. It is a combination of Lemma 5.10 and Proposition 5.11 below.

Corollary **C**. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid as in Theorem* **A**. *If $(A, \mathcal{H})$ is a GT Hopf algebroid, then for any object $x \in \mathscr{H}_0(\Bbbk)$, the comodule algebra (P_x, α_x) is a right principal $\mathcal{H}_x$-bundle (i.e. a Hopf Galois extension).*

Conversely, let (P, α) be a right principal B-bundle over a Hopf $\Bbbk$-algebra B with extension $\alpha : A \to P$. Denote by $\upsilon : \mathcal{H} := (P \otimes P)^{coinv_B} \to P \otimes P$ the canonical injection where $P \otimes P$ is a right B-comodule algebra via the diagonal coaction (here R^{coinv_B} denotes the subalgebra of coinvariant elements of a right B-comodule algebra R). Assume that υ is a faithfully flat extension and that

$$\mathcal{H} \otimes_A \mathcal{H} = \Big((P \otimes P) \otimes_A (P \otimes P)\Big)^{coinv_B},$$

where $(P \otimes P) \otimes_A (P \otimes P)$ is endowed within a canonical right B-comodule algebra structure. Then the pair of algebras $(A, \mathcal{H})$ admits a unique structure of GT Hopf algebroid such that $(\alpha, \upsilon) : (A, \mathcal{H}) \to (P, P \otimes P)$ is a morphism of GT Hopf algebroids.

2. Abstract groupoids: General notions and basic properties

This section contains the results about groupoids, which we want to transfer to the context of Hopf algebroids in the forthcoming sections. For sake of completeness we include some of their proofs.

2.1. Notations, basic notions and examples.

A *groupoid (or abstract groupoid)* is a small category where each morphism is an isomorphism. That is, a pair of two sets $\mathscr{G} := (G_1, G_0)$ with diagram $G_1 \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} G_0$, where s and t are resp. the source and the target of a given arrow, and ι assigns to each object its identity arrow; together with an associative and unital multiplication $G_2 := G_1 \, {}_{\mathsf{s}}\times_{\mathsf{t}} \, G_1 \to G_1$ as well as a map $G_1 \to G_1$ which associated to each arrow its inverse.

Given a groupoid $\mathscr{G}$, consider an object $x \in G_0$, *the isotropy group of $\mathscr{G}$ at x*, is the group:

$$\mathscr{G}^x := \Big\{ g \in G_1 | \, \mathsf{s}(g) = \mathsf{t}(g) = x \Big\}. \tag{1}$$

Notice that the disjoint union $\biguplus_{x \in G_0} \mathscr{G}^x$ of all isotropy groups form the set of arrows of a subgroupoid of $\mathscr{G}$ whose source equal to its target, namely, the projection $\biguplus_{x \in G_0} \mathscr{G}^x \to G_0$.

A *morphism of groupoids* $\phi : \mathscr{H} \to \mathscr{G}$ is a functor between the underlying categories. Obviously any such a morphism induces morphisms between the isotropy groups: $\phi^u : \mathscr{H}^u \to \mathscr{G}^{\phi_0(u)}$, for every $u \in H_0$. Naturally, groupoids, morphism of groupoids, and natural transformations form the 2-category Grpds.

Next we describe some typical examples of groupoids.

Example 2.1. A part from the classical example of the Poincaré groupoid of a topological space, or the groupoid associated to an atlas of a manifold etc., here is one other (formally) classical example. Let G be a group and fix a set M. Denote by BG_M the category whose objects are (left) G-torsors of the form (P, G, M) and morphisms are G-morphisms. In the terminology of Definition 2.8 below, an object (P, G, M) in BG_M is a *principal left G-set P*. Precisely, this is a left G-set P with projection $\pi : P \to M$ to the set of orbits M and where the canonical map $G \times P \to P\,{}_{\pi}\!\times_{\pi} P$, $(g, p) \mapsto (gp, p)$ is bijective. It is clear that any morphism in this category is an isomorphism, thus, BG_M is a groupoid (probably not small). The groupoid BG_{pt} plays a crucial role in the representation theory of the group G. Furthermore, when M varies in the category Sets of sets, we obtain the presheaf of groupoids $BG : \mathsf{Sets}^{op} \longrightarrow \mathsf{Grpds}$, which is known as *the classifying stack* of the group G.

Example 2.2. We present here several examples ordered by inclusion.

(1) One can associated to a given set X the so called *the groupoid of pairs* (called *fine groupoid* in [3] and *simplicial groupoid* in [15]), its set of arrows is defined by $G_1 = X \times X$ and the set of objects by $G_0 = X$; the sourse and the target are $\mathsf{s} = pr_2$ and $\mathsf{t} = pr_1$, the second and the first projections, and the map of identity arrows is ι the diagonal map. The multiplication and the inverse maps are given by
$$(x, x')\,(x', x'') = (x, x''), \quad \text{and} \quad (x, x')^{-1} = (x', x).$$
(2) Let $\mathsf{v} : X \to Y$ be a map. Consider the fibre product $X\,{}_{\mathsf{v}}\!\times_{\mathsf{v}} X$ as a set of arrows of the groupoid $X\,{}_{\mathsf{v}}\!\times_{\mathsf{v}} X \overset{pr_2}{\underset{pr_1}{\rightrightarrows}} X$, where as before $\mathsf{s} = pr_2$ and $\mathsf{t} = pr_1$, and the map of identity arrows is ι the diagonal map. The multiplication and the inverse are clear.
(3) Assume that $\mathcal{R} \subseteq X \times X$ is an equivalence relation on the set X. One can construct a groupoid $\mathcal{R} \overset{pr_2}{\underset{pr_1}{\rightrightarrows}} X$, with structure maps as before. This is an important class of groupoids known as *the groupoid of equivalence relation*, see [10, Exemple 1.4, page 301].

Example 2.3. Any group G can be considered as a groupoid by taking $G_1 = G$ and $G_0 = \{*\}$ (a set with one element). Now if X is a right G-set with action $\rho : X \times G \to X$, then one can define the so called *the action groupoid*: $G_1 = X \times G$ and $G_0 = X$, the source and the target are $\mathsf{s} = \rho$ and $\mathsf{t} = pr_1$, the identity map sends $x \mapsto (e, x) = \iota_x$, where e is the identity element of G. The multiplication is given by $(x, g)(x', g') = (x, gg')$, whenever $xg = x'$, and the inverse is defined by $(x, g)^{-1} = (xg, g^{-1})$. Clearly the pair of maps $(pr_2, *) : (G_1, G_0) \to (G, \{*\})$ defines a morphism of groupoids.

A more interesting for our purposes, and a bit elaborated example, is the following.

Example 2.4. Let $\mathscr{G} = (G_1, G_0)$ be a groupoid and $\varsigma : X \to G_0$ a map. Consider the following pair of sets:
$$G^{\varsigma}{}_1 := X\,{}_{\varsigma}\!\times_{\mathsf{t}} G_1\,{}_{\mathsf{s}}\!\times_{\varsigma} X = \Big\{(x, g, x') \in X \times G_1 \times X \,|\; \varsigma(x) = \mathsf{t}(g), \varsigma(x') = \mathsf{s}(g)\Big\}, \quad G^{\varsigma}{}_0 := X.$$
Then $\mathscr{G}^{\varsigma} = (G^{\varsigma}{}_1, G^{\varsigma}{}_0)$ is a groupoid, with structure maps: $\mathsf{s} = pr_3$, $\mathsf{t} = pr_1$, $\iota_x = (\varsigma(x), \iota_{\varsigma(x)}, \varsigma(x))$, $x \in X$. The multiplication is defined by $(x, g, y)(x', g', y') = (x, gg', y')$, whenever $y = x'$, and the inverse is given by $(x, g, y)^{-1} = (y, g^{-1}, x)$. The groupoid $\mathscr{G}^{\varsigma}$ is known as *the induced groupoid of $\mathscr{G}$ by the map ς*, (or *the pull-back groupoid of $\mathscr{G}$ along ς*, see [15] for dual notion). Clearly, there is a canonical morphism $\phi^{\varsigma} := (pr_2, \varsigma) : \mathscr{G}^{\varsigma} \to \mathscr{G}$ of groupoids.

Any morphism $\phi : \mathscr{H} \to \mathscr{G}$ of groupoids factors through the canonical morphism $\mathscr{G}^{\phi_0} \to \mathscr{G}$, that is we have the following (strict) commutative diagram
$$\begin{tikzcd} \mathscr{H} \arrow[rr, "\phi"] \arrow[dr, dashed, "\phi'"'] & & \mathscr{G} \\ & \mathscr{G}^{\phi_0} \arrow[ur] & \end{tikzcd}$$
of groupoids, where $\phi'_0 = id_{\mathscr{H}_0}$ and
$$\phi'_1 : H_1 \longrightarrow G^{\phi_0}{}_1, \quad \Big(h \longmapsto (\mathsf{t}(h), \phi_1(h), \mathsf{s}(h))\Big).$$
A particular and important example of an induced groupoid is the case when $\mathscr{G}$ is a groupoid with one object, that is, a group. In this case, to any group G and a set X, one can associated the groupoid $(X \times G \times X, X)$, as the induced groupoid of $(G, \{*\})$ by the map $* : X \to \{*\}$.

To repeat, a groupoid $\mathscr{G} = (G_1, G_0)$ is said to be *transitive* if the map $(\mathsf{s}, \mathsf{t}) : G_1 \to G_0 \times G_0$ is surjective.

Example 2.5. The groupoid of pair is clearly transitive, as well as any induced groupoid of the form $(X \times G \times X, X)$. On other hand, if a group G acts transitively on a set X, then the associated action groupoid is by construction transitive. Explicitly, let X be a right G-set whose G-action is transitive, that is, the set of orbits is a one element set. Thus, for every pair of elements $x, y \in X$ there exists $g \in G$ such that $y = xg$. This means that the map $X \times G \to X \times X$, $(x, g) \mapsto (x, xg)$ is a surjective, and so the action groupoid $(X \times G, X)$ is transitive.

It is clear that in a transitive groupoid there is, up to isomorphisms, only one type of isotropy groups. In other words, any two isotropy groups are conjugated, equivalently, the groupoid have only one connected component. This means that, for any two distinct objects $x, y \in G_0$, there is a non-canonical isomorphism of groups $\mathscr{G}^x \cong \mathscr{G}^y$ given by conjugation: Let $g \in G_1$ with $x = \mathsf{s}(g)$ and $\mathsf{t}(g) = y$, then

$$\mathscr{G}^x \longrightarrow \mathscr{G}^y, \Big(h \longmapsto ghg^{-1}\Big)$$

is an isomorphism of groups. This fact is essential in showing that any transitive groupoid is isomorphic, in a non-canonical way, to an induced groupoid of the form $(X \times G \times X, X)$. Indeed, given a transitive groupoid $\mathscr{G}$, fix an object $x \in G_0$ with isotropy group $\mathscr{G}^x$ and chose a family of arrows $\{f_y\}_{y \in G_0}$ such that $f_y \in \mathsf{t}^{-1}(\{x\})$ and $\mathsf{s}(f_y) = y$, for $y \neq x$ while $f_x = \iota(x)$, for $y = x$. In this way the morphism

$$\phi^x : \mathscr{G} \xrightarrow{\cong} (G_0 \times \mathscr{G}^x \times G_0, G_0), \qquad \Big((g, z) \longmapsto \Big((\mathsf{s}(g),\, f_{\mathsf{t}(g)}\, g\, f_{\mathsf{s}(g)}^{-1}, \mathsf{t}(g)), z\Big)\Big)$$

establishes an isomorphism of groupoids.

2.2. **Groupoids actions, equivariant maps and the orbits sets.** The following definition is a natural generalization to the context of groupoids, of the usual notion of group-set. It is an abstract formulation of that given in [20, Definition 1.6.1] for Lie groupoids, and essentially the same definition based on the Sets-bundles notion given in [26, Definition 1.11].

Definition 2.6. Given a groupoid $\mathscr{G}$ and a map $\varsigma : X \to G_0$. We say that (X, ς) is a *right $\mathscr{G}$-set*, if there is a map (*the action*) $\rho : X\, {}_{\varsigma}\!\times_{\mathsf{t}}\, G_1 \to X$ sending $(x, g) \mapsto xg$, satisfying the following conditions

1. $\mathsf{s}(g) = \varsigma(xg)$, for any $x \in X$ and $g \in G_1$ with $\varsigma(x) = \mathsf{t}(g)$.
2. $x\iota_{\varsigma(x)} = x$, for every $x \in X$.
3. $(xg)h = x(gh)$, for every $x \in X$, $g, h \in G_1$ with $\varsigma(x) = \mathsf{t}(g)$ and $\mathsf{t}(h) = \mathsf{s}(g)$.

A *left action* is analogously defined by interchanging the source with the target. Obviously, any groupoid $\mathscr{G}$ acts over itself on both sides by using the regular action, i.e. the multiplication $G_1\, {}_{\mathsf{s}}\!\times_{\mathsf{t}}\, G_1 \to G_1$. That is, (G_1, s) is a right $\mathscr{G}$-set and (G_1, t) is a left $\mathscr{G}$-set with this action.

Let (X, ς) be a right $\mathscr{G}$-set, and consider the pair of sets $X \rtimes \mathscr{G} := \Big(X\, {}_{\varsigma}\!\times_{\mathsf{t}}\, G_1, X\Big)$ as a groupoid with structure maps $\mathsf{s} = \rho$, $\mathsf{t} = pr_1$, $\iota_x = (x, \iota_{\varsigma(x)})$. The multiplication and the inverse maps are defined by $(x, g)(x', g') = (x, gg')$ and $(x, g)^{-1} = (xg, g^{-1})$. The groupoid $X \rtimes \mathscr{G}$ is known as the *right translation groupoid of X by $\mathscr{G}$*.

For sake of completeness let us recall the notion of equivariant maps. A *morphism of right $\mathscr{G}$-sets* (or *$\mathscr{G}$-equivariant map*) $F : (X, \varsigma) \to (X', \varsigma')$ is a map $F : X \to X'$ such that the diagrams

$$\begin{tikzcd} & X \arrow[dl, "\varsigma"'] \arrow[dd, "F"] \\ G_0 & \\ & X' \arrow[ul, "\varsigma'"] \end{tikzcd} \qquad \begin{tikzcd} X\, {}_{\varsigma}\!\times_{\mathsf{t}}\, G_1 \arrow[r] \arrow[d, "F \times id"'] & X \arrow[d, "F"] \\ X'\, {}_{\varsigma'}\!\times_{\mathsf{t}}\, G_1 \arrow[r] & X' \end{tikzcd} \tag{2}$$

commute. Clearly any such a $\mathscr{G}$-equivariant map induces a morphism of groupoids $\mathsf{F} : X \rtimes \mathscr{G} \to X' \rtimes \mathscr{G}$.

Next we recall the notion of the orbit set attached to a right groupoid-set. This notion is a generalization of the orbit set in the context of group-sets. Here we use the (right) translation groupoid to introduce this set. First we recall the notion of the orbit set of a given groupoid. *The orbit set of a groupoid $\mathscr{G}$* is the quotient set of G_0 by the following equivalence relation: take an object $x \in G_0$, define

$$\mathscr{O}_x := \mathsf{t}(\mathsf{s}^{-1}(x)) \;=\; \Big\{y \in G_0 |\; \exists\, g \in G_1 \text{ such that } \mathsf{s}(g) = x,\ \mathsf{t}(g) = y\Big\}, \tag{3}$$

which is equal to the set $\mathsf{s}(\mathsf{t}^{-1}(x))$. This is a non empty set, since $x \in \mathscr{O}_x$. Two objects $x, x' \in G_0$ are said to be equivalent if and only if $\mathscr{O}_x = \mathscr{O}_{x'}$, or equivalently, two objects are equivalent if and only if there is an arrow connecting them. This in fact defines an equivalence relation whose quotient set is denoted by $G_0/\mathscr{G}$. In others words, this is the set of all connected components of $\mathscr{G}$.

Given a right $\mathscr{G}$-set (X, ς), the *orbit set* $X/\mathscr{G}$ of (X, ς) is the orbit set of the translation groupoid $X \rtimes \mathscr{G}$. If $\mathscr{G} = (X \times G, X)$ is an action groupoid as in Example 2.3, then obviously the orbit set of this groupoid coincides with the classical set X/G of orbits of X.

2.3. **Principal groupoid-bisets and the two sided translation groupoid.** Let $\mathscr{G}$ and $\mathscr{H}$ be two groupoids and $(X, \varsigma, \vartheta)$ a triple consisting of a set X and two maps $\varsigma : X \to G_0$, $\vartheta : X \to H_0$. The following definitions are abstract formulations of those given in [17, 21] for topological and Lie groupoids.

Definition 2.7. The triple $(X, \varsigma, \vartheta)$ is said to be an $(\mathscr{H}, \mathscr{G})$-*biset* if there is a left $\mathscr{H}$-action $\lambda : H_1\, {}_{\mathsf{s}}\!\times_{\vartheta} X \to X$ and right $\mathscr{G}$-action $\rho : X\, {}_{\varsigma}\!\times_{\mathsf{t}} G_1 \to X$ such that

(1) For any $x \in X$, $h \in H_1$, $g \in G_1$ with $\vartheta(x) = \mathsf{s}(h)$ and $\varsigma(x) = \mathsf{t}(g)$, we have
$$\vartheta(xg) = \vartheta(x) \;\text{ and }\; \varsigma(hx) = \varsigma(x).$$
(2) For any $x \in X$, $h \in H_1$ and $g \in G_1$ with $\varsigma(x) = \mathsf{t}(g)$, $\vartheta(x) = \mathsf{s}(h)$, we have $h(xg) = (hx)g$.

The two sided translation groupoid associated to a given $(\mathscr{H}, \mathscr{G})$-biset $(X, \varsigma, \vartheta)$ is defined to be the groupoid $\mathscr{H} \ltimes X \rtimes \mathscr{G}$ whose set of objects is X and set of arrows is
$$H_1\, {}_{\mathsf{s}}\!\times_{\vartheta} X\, {}_{\varsigma}\!\times_{\mathsf{s}} G_1 = \Big\{(h, x, g) \in H_1 \times X \times G_1 |\ \mathsf{s}(h) = \vartheta(x),\ \mathsf{s}(g) = \varsigma(x)\Big\}.$$
The structure maps are:
$$\mathsf{s}(h, x, g) = x, \quad \mathsf{t}(h, x, g) = hxg^{-1} \quad \text{and} \quad \iota_x = (\iota_{\vartheta(x)}, x, \iota_{\varsigma(x)}).$$
The multiplication and the inverse are given by:
$$(h, x, g)(h', x', g') = (hh', x', gg'), \quad (h, x, g)^{-1} = (h^{-1}, hxg^{-1}, g^{-1}).$$
Associated to a given $(\mathscr{H}, \mathscr{G})$-biset $(X, \varsigma, \vartheta)$, there are two canonical morphism of groupoids:
$$\Sigma : \mathscr{H} \ltimes X \rtimes \mathscr{G} \longrightarrow \mathscr{G}, \qquad \Big((h, x, g), y\Big) \longmapsto \Big(g, \varsigma(y)\Big), \tag{4}$$
$$\Theta : \mathscr{H} \ltimes X \rtimes \mathscr{G} \longrightarrow \mathscr{H}, \qquad \Big((h, x, g), y\Big) \longmapsto \Big(h, \vartheta(y)\Big). \tag{5}$$

Definition 2.8. Let $(X, \varsigma, \vartheta)$ be an $(\mathscr{H}, \mathscr{G})$-biset. We say that $(X, \varsigma, \vartheta)$ is a *left principal* $(\mathscr{H}, \mathscr{G})$-*biset* if it satisfies the following conditions:

(P-1) $\varsigma : X \to G_0$ is surjective;
(P-2) the canonical map
$$\nabla : H_1\, {}_{\mathsf{s}}\!\times_{\vartheta} X \longrightarrow X\, {}_{\varsigma}\!\times_{\varsigma} X, \quad \Big((h, x) \longmapsto (hx, x)\Big) \tag{6}$$
is bijective.

By condition (P-2) we consider the map $\delta := pr_1 \circ \nabla^{-1} : X\, {}_{\varsigma}\!\times_{\varsigma} X \to H_1$. This map clearly satisfies:
$$\mathsf{s}(\delta(x, y)) = \vartheta(y) \tag{7}$$
$$\delta(x, y)y = x, \quad \text{for any } x, y \in X \text{ with } \varsigma(x) = \varsigma(y); \tag{8}$$
$$\delta(hx, x) = h, \quad \text{for } h \in H_1,\ x \in X \text{ with } \mathsf{s}(h) = \vartheta(x). \tag{9}$$
Equation (9), shows that the action is in fact free, that is, $hx = x$ only when $h = \iota_{\vartheta(x)}$. The subsequent lemma is also immediate from this definition.

Lemma 2.9. *Let $(X, \varsigma, \vartheta)$ be a left principal $(\mathscr{H}, \mathscr{G})$-biset. Then the map ς induces a bijection between the orbit set $X/\mathscr{H}$ and the set of objects G_0.*

Analogously one defines *right principal* $(\mathscr{H}, \mathscr{G})$-*biset*. *A principal* $(\mathscr{H}, \mathscr{G})$-*biset* is both left and right principal biset. For instance, $(G_1, \mathsf{t}, \mathsf{s})$ is a left and right principal $(\mathscr{G}, \mathscr{G})$-biset, known as the *unit principal biset*, which we denote by $\mathscr{U}(\mathscr{G})$. More examples of left principal biset can be performed, as in the geometric case, by pulling back other left principal biset. Precisely, assume we are given $(X, \varsigma, \vartheta)$ a left principal $(\mathscr{H}, \mathscr{G})$-biset, and let $\psi : \mathscr{K} \to \mathscr{G}$ be a morphism of groupoids. Consider the set $Y := X\, {}_{\vartheta}\!\times_{\psi_0} K_0$

together with the maps $pr_2 : Y \to K_0$ and $\widetilde{\varsigma} := \varsigma \circ pr_1 : Y \to H_0$. Then the triple $(Y, \widetilde{\varsigma}, pr_2)$ is an $(\mathscr{H}, \mathscr{K})$-biset with actions

$$\lambda : H_1\, {}_{\mathsf{s}}\!\times_{\varphi}\, Y \longrightarrow Y, \qquad (h,(x,u)) \longmapsto (hx,u) \tag{10}$$

$$\rho : Y\, {}_{\widetilde{\varphi}}\!\times_{\mathsf{t}}\, K_1 \longrightarrow Y, \qquad ((x,u),f) \longmapsto (x\psi_1(f), \mathsf{s}(f)), \tag{11}$$

which is actually a left principal $(\mathscr{H}, \mathscr{K})$-biset, and known as the *pull-back principal biset of* $(X, \varsigma, \vartheta)$; we denote it by $\psi^*((X,\varsigma,\vartheta))$. A left principal biset is called a *trivial left principal biset* if it is the pull-back of the unit left principal biset, that is, of the form $\psi^*(\mathscr{U}(\mathscr{G}))$ for some morphism of groupoids $\psi : \mathscr{K} \to \mathscr{G}$.

Next we expound the bicategorical constructions beyond the notion of principal groupoids-bisets. A *morphism of left principal* $(\mathscr{H},\mathscr{G})$*-bisets* $F : (X,\varsigma,\vartheta) \to (X',\varsigma',\vartheta')$ is a map $F : X \to X'$ which is simultaneously $\mathscr{G}$-equivariant and $\mathscr{H}$-equivariant, that is, the following diagrams

$$\begin{tikzcd} & X \arrow[dl, "\varsigma"'] \arrow[dd, "F"] \arrow[dr, "\vartheta"] & \\ G_0 & & H_0 \\ & X' \arrow[ul, "\varsigma'"] \arrow[ur, "\vartheta'"'] & \end{tikzcd} \qquad \begin{tikzcd} X\, {}_{\varsigma}\!\times_{\mathsf{t}}\, G_1 \arrow[r] \arrow[d, "F\times id"'] & X \arrow[d, "F"] \\ X'\, {}_{\varsigma'}\!\times_{\mathsf{t}}\, G_1 \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} H_1\, {}_{\mathsf{s}}\!\times_{\vartheta}\, X \arrow[r] \arrow[d, "id\times F"'] & X \arrow[d, "F"] \\ H_1\, {}_{\mathsf{s}}\!\times_{\vartheta'}\, X' \arrow[r] & X'. \end{tikzcd} \tag{12}$$

commute. *An isomorphism of left principal bisets* is a morphism whose underlying map is bijective. As in the geometric case we have:

Proposition 2.10. *Given two groupoids $\mathscr{G}$ and $\mathscr{H}$. Then any morphism between left principal $(\mathscr{H},\mathscr{G})$-bisets is an isomorphism.*

Proof. Let $F : (X,\varsigma,\vartheta) \to (X',\varsigma',\vartheta')$ be a morphism of left principal $(\mathscr{H},\mathscr{G})$-biset. We first show that F is injective. So take $x, y \in X$ such that $F(x) = F(y)$, whence $\varsigma(x) = \varsigma(y)$. By Lemma 2.9, we know that there exists $h \in H_1$ with $\mathsf{s}(h) = \vartheta(x)$ such that $hx = y$. Therefore, we have $F(hx) = F(y) = hF(x) = F(x)$ and so $h = \iota_{\vartheta(x)}$, since the left action is free. This shows that $x = y$. The surjectivity of F is derived as follows. Take an arbitrary element $x' \in X'$ and consider its image $\varsigma'(x') \in G_0$. Since ς is surjective, there exists $x \in X$ such that $\varsigma(x) = \varsigma' F(x) = \varsigma'(x')$. This means that $F(x)$ and x' are in the same orbit, so there exists $h' \in H_1$ (with $\mathsf{s}(h') = \vartheta(x)$) such that $h'F(x) = F(h'x) = x'$, which shows that F is surjective. □

Remark 2.11. By Proposition 2.10, the category of left principal bisets $\mathsf{PB}^l(\mathscr{H},\mathscr{G})$ is actually a groupoid (not necessarily a small category). On the other hand, notice that if (X,ς,ϑ) is a left principal $(\mathscr{H},\mathscr{G})$-biset, then its opposite $(X^o, \vartheta, \varsigma)$ is a right principal $(\mathscr{G},\mathscr{H})$-biset, where the underlying set still the same set X while the actions were switched by using the inverse maps of both groupoids. This in fact establishes an isomorphism of categories between $\mathsf{PB}^l(\mathscr{H},\mathscr{G})$ and the category of right principal biset $\mathsf{PB}^r(\mathscr{G},\mathscr{H})$.

Remark 2.12. Given (X,ς,ϑ) an $(\mathscr{H},\mathscr{G})$-biset and $(X',\varsigma',\vartheta')$ a $(\mathscr{G},\mathscr{K})$-biset. One can endow the fibre product $X\, {}_{\vartheta}\!\times_{\vartheta'}\, X'$ within a structure of an $(\mathscr{H},\mathscr{K})$-biset. Furthermore, $\mathscr{G}$ also acts on this set by the action $(x,x').g = (xg, g^{-1}x')$, for $g \in G_1$, $(x,x') \in X\, {}_{\vartheta}\!\times_{\vartheta'}\, X'$ with $\mathsf{t}(g) = \vartheta(x) = \vartheta'(x')$. Denote by $X \otimes_{\mathscr{G}} X' := (X\, {}_{\vartheta}\!\times_{\vartheta'}\, X')/\mathscr{G}$ its orbit set, then clearly this set inherits a structure of $(\mathscr{H},\mathscr{K})$-biset. This is *the tensor product of bisets*, also known as *le produit contracté* [14, Définition 1.3.1 page 114], [10, Chap.III, §4, 3.1]. It turns out that, if (X,ς,ϑ) is a left principal biset and $(X',\varsigma',\vartheta')$ is a left principal biset, then $X \otimes_{\mathscr{G}} X'$ is a left principal $(\mathscr{H},\mathscr{K})$-biset. Moreover, one can show that the tensor product (over different groupoids) is associative, up to a natural isomorphism. This defines the bicategory PB^l of left principal bisets. Analougly, we have the bicategories PB^r and PB^b (of principal bisets).

For a single 0-cell, that is, a groupoid $\mathscr{G}$, the category $\mathsf{PB}^b(\mathscr{G},\mathscr{G})$ turns to be a *bigroup* (or *a categorical group*). Moreover, in analogy with the group case, one can construct with the help of Proposition 2.10 and by using morphisms between left translation groupoids, a presheaf $\mathscr{B}\mathscr{G} : \mathsf{Sets}^{op} \longrightarrow \mathsf{2\text{-}Grpds}$ to the category of 2-groupoids known as *the classifying 2-stack* of the groupoid $\mathscr{G}$.

2.4. **Principal groupoids-biset versus weak equivalences.** A morphism of groupoids $\phi : \mathscr{H} \to \mathscr{G}$ is said to be a *weak equivalence* if it satisfies the following two conditions:

(WE-1) The composition map $G_1\, {}_{\mathsf{s}}\!\times_{\phi_0}\, H_0 \xrightarrow{pr_1} G_1 \xrightarrow{\mathsf{t}} G_0$ is surjective, or equivalently, the composition map $H_0\, {}_{\phi_0}\!\times_{\mathsf{t}}\, G_1 \xrightarrow{pr_2} G_1 \xrightarrow{\mathsf{s}} G_0$ is surjective.

(WE-2) The following diagram is cartesian

$$\begin{tikzcd} H_1 \arrow[r, "\phi_1"] \arrow[d, "(\mathsf{s},\mathsf{t})"'] & G_1 \arrow[d, "(\mathsf{s},\mathsf{t})"] \\ H_0 \times H_0 \arrow[r, "\phi_0\times\phi_0"] & G_0 \times G_0 \end{tikzcd}$$

Equivalently there is a bijection $\Gamma : H_1 \cong H_0 \,{}_{\phi_0}\!\times_{\mathsf{s}} G_1 \,{}_{\mathsf{t}}\!\times_{\phi_0} H_0$ such that $pr_2 \circ \Gamma = \phi_0$ and $(pr_1,\ pr_3) \circ \Gamma = (\mathsf{s}, \mathsf{t})$, in the sense that the diagram

$$\begin{tikzcd} H_1 \arrow[dr, "\cong", "\Gamma"'] \arrow[drrr, bend left, "\phi_1"] \arrow[ddr, bend right, "(\mathsf{s},\mathsf{t})"'] & & & \\ & H_0 \,{}_{\phi_0}\!\times_{\mathsf{s}} G_1 \,{}_{\mathsf{t}}\!\times_{\phi_0} H_0 \arrow[rr, "pr_2"] \arrow[d, "(pr_1, pr_3)"'] & & G_1 \arrow[d, "(\mathsf{s},\mathsf{t})"] \\ & H_0 \times H_0 \arrow[rr, "\phi_0\times\phi_0"] & & G_0 \times G_0 \end{tikzcd}$$

commutes.

In categorical terms, condition (WE-1) says that ϕ is an *essentially surjective* functor: Each object of $\mathscr{G}$ is isomorphic to the image by ϕ of an object in $\mathscr{H}$. The second condition, means that ϕ is *fully faithful*: If u, v are two objects in $\mathscr{H}$ then ϕ defines a bijection between the sets of arrows $\mathscr{H}(u, v)$ and $\mathscr{G}(\phi_0(u), \phi_0(v))$. Both properties classically characterize functors which define equivalences of categories.

Two groupoids $\mathscr{G}$ and $\mathscr{H}$ are said to be *weakly equivalent* when there exists a third groupoid $\mathscr{K}$ with a diagram (i.e. a *span*) of weak equivalences:

$$\begin{tikzcd} & \mathscr{K} \arrow[dl] \arrow[dr] & \\ \mathscr{G} & & \mathscr{H}. \end{tikzcd}$$

For sake of completeness, next we give a result which relate the notion of principal biset with that of weak equivalence.

Proposition 2.13. *Let $\mathscr{G}$ and $\mathscr{H}$ be two groupoids. Assume that there is $(X, \varsigma, \vartheta)$ a principal $(\mathscr{H}, \mathscr{G})$-biset. Then the canonical morphisms of groupoids*

$$\begin{tikzcd} & \mathscr{H} \ltimes X \rtimes \mathscr{G} \arrow[dl, "\Theta"'] \arrow[dr, "\Sigma"] & \\ \mathscr{H} & & \mathscr{G} \end{tikzcd}$$

are weak equivalences, where Θ, Σ are as in (4) *and* (5). *In particular, $\mathscr{G}$ and $\mathscr{H}$ are weakly equivalent.*

Proof. We only show that if $(X, \varsigma, \vartheta)$ is a left principal $(\mathscr{H}, \mathscr{G})$-biset then the canonical morphism

$$\Sigma : \mathscr{H} \ltimes X \rtimes \mathscr{G} \longrightarrow \mathscr{G}, \quad \Big(((h, x, g), x) \longmapsto (g, \varsigma(x)\Big)$$

is a weak equivalence. The proof of the fact that Θ is a weak equivalence follows similarly from the assumption that $(X, \varsigma, \vartheta)$ is right principal $(\mathscr{H}, \mathscr{G})$-biset.

Condition (WE-1) for Σ is clear, since ς is surjective by condition (P-1). Consider the map

$$\begin{aligned} \Gamma : H_1 \,{}_{\mathsf{s}}\!\times_{\vartheta} X \,{}_{\varsigma}\!\times_{\mathsf{s}} G_1 &\longrightarrow X \,{}_{\varsigma}\!\times_{\mathsf{s}} G_1 \,{}_{\mathsf{t}}\!\times_{\varsigma} X \\ (h, x, g) &\longmapsto (x, g, hxg^{-1}). \end{aligned}$$

Using the map $\delta : X \,{}_{\varsigma}\!\times_{\varsigma} X \to H_1$ resulting from condition (P-2) on $(X, \varsigma, \vartheta)$ and which satisfies equations (7)-(9), we define the inverse of Γ to be the map:

$$\begin{aligned} \Gamma^{-1} : X \,{}_{\varsigma}\!\times_{\mathsf{s}} G_1 \,{}_{\mathsf{t}}\!\times_{\varsigma} X &\longrightarrow H_1 \,{}_{\mathsf{s}}\!\times_{\vartheta} X \,{}_{\varsigma}\!\times_{\mathsf{s}} G_1 \\ (x, g, y) &\longmapsto (\delta(y, xg^{-1}), x, g), \end{aligned}$$

which gives condition (WE-2) for Σ. □

Remark 2.14. As we have seen in Remark 2.11, the opposite of left principal $(\mathscr{H},\mathscr{G})$-biset is a right principal $(\mathscr{G},\mathscr{H})$-biset. Thus the opposite of principal biset is also a principal biset. In this way, Proposition 2.13 says that the "equivalence relation" between groupoids defined by '*being connected by a principal biset*' is contained in the equivalence relation defined by '*weakly equivalent*'. An interesting question is then to check if both relations are the same. Precisely, one can ask whether two weakly equivalent groupoids $\mathscr{H}$ and $\mathscr{G}$ are connected by a certain principal $(\mathscr{H},\mathscr{G})$-biset. As was shown in [11], in the framework of commutative flat Hopf algebroids, the answer to this question is positive, see Remark 3.8.

2.5. **Transitive groupoids are characterized by weak equivalences.** This subsection is the main motivation for the forthcoming sections. Here we show perhaps a well known result that characterizes transitive groupoids by means of weak equivalences and principal groupoids-bisets.

Proposition 2.15. *Let $\mathscr{G}$ be an abstract groupoid. Then the following are equivalent:*

(i) For every map $\varsigma : X \to G_0$, the induced morphism of groupoids $\phi^{\varsigma} : \mathscr{G}^{\varsigma} \to \mathscr{G}$ is a weak equivalence;
(ii) $\mathscr{G}$ is a transitive groupoid;
(iii) For every map $\varsigma : X \to G_0$, the pull-back biset ${\phi^{\varsigma}}^*(\mathscr{U}(\mathscr{G}))$ is a principal $(\mathscr{G},\mathscr{G}^{\varsigma})$-biset.

Proof. $(i) \Rightarrow (ii)$. Is immediate.

$(ii) \Rightarrow (iii)$. By definition ${\phi^{\varsigma}}^*(\mathscr{U}(\mathscr{G}))$ is a left principal $(\mathscr{G},\mathscr{G}^{\varsigma})$-biset. We need then to check that, under condition (ii), it is also right principal $(\mathscr{G},\mathscr{G}^{\varsigma})$-biset. This biset is given by ${\phi^{\varsigma}}^*(\mathscr{U}(\mathscr{G})) = (G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X, \widetilde{\mathsf{t}}, pr_2)$, where $\widetilde{\mathsf{t}} := \mathsf{t}\circ pr_1 : G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X \to G_1 \to G_0$. The left and right actions are given as in equations (10) and (11) by

$$g \rightharpoonup (f,x) \,=\, (gf,x) \quad \text{and} \quad (f,x) \leftharpoonup (y,h,x) \,=\, (fh,y),$$

for any $(f,x) \in G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X$, $g \in G_1$ with $\mathsf{s}(g) = \mathsf{t}(f)$, and $h \in G_1$ with $\mathsf{s}(h) = \varsigma(y)$, $\mathsf{t}(h) = \varsigma(x)$ and $\mathsf{s}(f) = \mathsf{t}(h)$. Both conditions (1)-(2) in Definition 2.7 are clearly satisfied.

The right canonical map is defined by

$$\nabla' : \Big(G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X\Big)\, {}_{pr_2}\!\times_{\mathsf{t}}\, G^{\varsigma}{}_1 \longrightarrow \Big(G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X\Big)\, {}_{\widetilde{\mathsf{t}}}\!\times_{\widetilde{\mathsf{t}}}\, \Big(G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X\Big), \ \ \Big((f,x),(y,h,x)\Big) \longmapsto \Big((f,x),(fh,y)\Big).$$

The map $\widetilde{\mathsf{t}}$ is clearly surjective, since $\mathscr{G}$ is transitive. This gives condition (P-1) for ${\phi^{\varsigma}}^*(\mathscr{U}(\mathscr{G}))$ as a right principal biset. Now, we need to check that ∇' is bijective, that is, condition (P-2) is fulfilled. However, the inverse of this map is easily shown to be the following map

$$\nabla'^{-1} : \Big(G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X\Big)\, {}_{\widetilde{\mathsf{t}}}\!\times_{\widetilde{\mathsf{t}}}\, \Big(G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X\Big) \longrightarrow \Big(G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X\Big)\, {}_{pr_2}\!\times_{\mathsf{t}}\, G^{\varsigma}{}_1, \ \ \Big((f,x),(f',x')\Big) \longmapsto \Big((f,x),(x',f^{-1}f',x)\Big).$$

$(iii) \Rightarrow (i)$. If we assume that ${\phi^{\varsigma}}^*(\mathscr{U}(\mathscr{G}))$ is a principal $(\mathscr{G},\mathscr{G}^{\varsigma})$-biset, then the map $\widetilde{\mathsf{t}}$ above, should be surjective. Therefore, the map

$$G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X \xrightarrow{pr_1} G_1 \xrightarrow{\mathsf{t}} G_0$$

is also surjective, which is condition (WE-1) for the morphism ϕ^{ς}. Condition (WE-2) for this morphism is trivial, since by definition we know that $G^{\varsigma}{}_1 = X\, {}_{\varsigma}\!\times_{\mathsf{t}}\, G_1\, {}_{\mathsf{s}}\!\times_{\varsigma}\, X$. □

2.6. **Correspondence between transitive groupoids and principal group-sets.** A particular example of principal groupoids-bisets are of course principal group-sets. As we will see below transitive groupoids are characterized by these group-sets. Precisely, there is a (non canonical) correspondence between transitive groupoids and principal group-sets.

Recall that a triple (P, G, π) consisting of a group G, a left G-set P, and a map $\pi : P \to G_0$, is said to be a *left principal G-set*, if the following conditions are satisfied:

(P'1) π is surjective;
(P'2) $\pi(gp) = \pi(p)$, for every $p \in P$ and $g \in G$;
(P'3) The canonical map $G \times P \longrightarrow P\, {}_{\pi}\!\times_{\pi}\, P$ sending $(g, p) \mapsto (gp, p)$ is bijective.

Equivalently, the action is free and G_0 is the orbit set.

Comparing with Definition 2.7, this means that the triple $(P, *, \pi)$, where $* : P \to \{*\}$, is a principal left (G, G_0)-biset, where the group G is considered as a groupoid with one object $\{*\}$ and G_0 is considered as a groupoid whose underlying category is a discrete category (i.e. category with only identities arrows) with set of objects G_0, and acts trivially on P.

In the previous situation, consider $P \times P$ as a left G-set by the diagonal action and denote by $G_1 := (P \times P)/G$ its set of orbits. The pair (G_1, G_0) admits as follows a structure of transitive groupoid.

Indeed, let $(p, p') \in P \times P$ and denote by $[(p, p')] \in (P \times P)/G$ its equivalence class. The source and target are $\mathsf{s}([(p, p')]) = \pi(p')$ and $\mathsf{t}([(p, p')]) = \pi(p)$. The identity arrow of an object $x \in G_0$ is given, using conditions (P'1)-(P'2), by the class $[(p, p)]$ where $\pi(p) = x$. Let $\mathfrak{p}, \mathfrak{q}$ be two equivalence classes in G_1 such that $\mathsf{s}(\mathfrak{p}) = \mathsf{t}(\mathfrak{q})$. Henceforth, if (p, p') is a representative of $\mathfrak{p}$, then $\mathfrak{q}$ can be represented by (p', p''). The multiplication $\mathfrak{p}\mathfrak{q}$ is then represented by (p, p''). This is a well defined multiplication since the action is free. By conditions (P'1), (P'3), we have that (G_1, G_0) is a transitive groupoid with a canonical morphism of groupoids

$$\begin{tikzcd} P \times P \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \arrow[l] \arrow[d] \\ G_1 := (P \times P)/G \arrow[r, shift left] \arrow[r, shift right] & P/G := G_0. \arrow[l] \end{tikzcd}$$

Conversely, given a transitive groupoid $\mathscr{G}$, and fix an object $x \in G_0$. Set $G := \mathscr{G}^x$ the isotropy group of x and let $P := \mathsf{t}^{-1}(\{x\})$ be the set of all arrows with target this x, i.e. the left star set of x. Consider the left G-action $G \times P \to P$ derived from the multiplication of $\mathscr{G}$. Since $\mathscr{G}$ is transitive, the triple (P, G, π) satisfies then the above conditions (P'1)-(P'3), which means that it is a left principal G-set.

We have then establish a (non-canonical) correspondence between transitive abstract groupoids and principal group-sets.

3. Hopf algebroids: comodules algebras, principal bundles, and weak equivalences

This section contains the definitions of commutative Hopf algebroids and theirs bicomodules algebras. All definitions are given in the algebraic way. Nevertheless, we will use a slightly superficial language of presheaves, sufficiently enough to make clearer the connection with the contents of Section 2.

Parallel to subsections 2.3 and 2.4 a brief contains on principal bibundles between Hopf algebroids and their connection with weak equivalences, is also presented. Dualizable objects in the category of (right) comodules are treated in the last subsection, where we also proof some useful lemmata.

3.1. **Preliminaries and basic notations.** We work over a commutative ground base field $\Bbbk$. Unadorned tensor product $- \otimes -$ stands for the tensor product of $\Bbbk$-vector spaces $- \otimes_{\Bbbk} -$. By $\Bbbk$-algebra, algebra or ring, we understood commutative $\Bbbk$-algebras, unless otherwise specified. The category of (right) A-modules over an algebra A, is denoted by Mod_A. The $\Bbbk$-vector space of all A-*linear maps* between two (right) A-modules M and N, is denoted by $\mathrm{Hom}_A(M, N)$. When $N = A$ is the regular module, we denote $M^* := \mathrm{Hom}_A(M, A)$.

Given two algebras R, S, we denote by $S(R) := \mathrm{Alg}_{\Bbbk}(S, R)$ the set of all $\Bbbk$-algebra maps from S to R. In what follows, a *presheaf* of sets (of groups, or of groupoids) stands for a functor from the opposite category of algebras $\mathrm{Alg}_{\Bbbk}^{op}$ to the category of sets (groups, or groupoids). Clearly, to any algebra A, there is an associated presheaf which sends $C \to A(C) = \mathrm{Alg}_{\Bbbk}(C, A)$, thus, the presheaf represented by A.

For two algebra maps $\sigma : A \to T$ and $\gamma : B \to T$ we denote by ${}_{\sigma}T_{\gamma}$ (respectively, ${}_{\sigma}T$ or T_{γ}, if one of the algebra map is the identity) the underlying (A, B)-bimodule of T (respectively, the underlying A-module of T) whose left A-action is induced by σ while its right B-action is induced by γ, that is,

$$a\,.\,t \,=\, \sigma(a)t, \quad t\,.\,b \,=\, t\gamma(b), \quad \text{for every } \ a \in A,\ b \in B,\ t \in T.$$

Assume there is an algebra map $x \in A(R)$. The extension functor $(-)_x : \mathsf{Mod}_A \to \mathsf{Mod}_R$ is the functor which sends any A-module M to the extended R-module $M_x = M \otimes_A R$. In order to distinguish between two extension functors, we use the notation $M_x := M \otimes_x R$ and $M_y := M \otimes_y S$, whenever another algebra map $y \in A(S)$ is given.

In the sequel we will use the terminology *coring* (or *cogébroïde* as in [9, 4]) for coalgebra with possibly different left-right structures on its underlying modules over the base ring. We refer to [6] for basic notions and properties of these objects.

3.2. **The 2-category of Hopf algebroids.** Recall from, *e.g.*, [25] that a *commutative Hopf algebroid*, or a *Hopf algebroid over a field* $\Bbbk$, is a pair $(A, \mathcal{H})$ of two commutative $\Bbbk$-algebras, together with algebra maps

$$\eta : A \otimes A \to \mathcal{H}, \quad \varepsilon : \mathcal{H} \to A, \quad \Delta : {}_{\mathsf{s}}\mathcal{H}_{\mathsf{t}} \to {}_{\mathsf{s}}\mathcal{H}_{\mathsf{t}} \otimes_A {}_{\mathsf{s}}\mathcal{H}_{\mathsf{t}}, \quad \mathscr{S} : {}_{\mathsf{s}}\mathcal{H}_{\mathsf{t}} \to {}_{\mathsf{t}}\mathcal{H}_{\mathsf{s}}$$

and a structure $({}_{\mathsf{s}}\mathcal{H}_{\mathsf{t}}, \Delta, \varepsilon)$ of an A-coring with $\mathscr{S}$ an A-coring map to the opposite coring. Therein the source and the target are the algebra maps $\mathsf{s} : A \to \mathcal{H}$ and $\mathsf{t} : A \to \mathcal{H}$ defined by $\mathsf{s}(a) = \eta(a \otimes 1)$ and $\mathsf{t}(a) = \eta(1 \otimes a)$, for every $a \in A$. The map $\mathscr{S}$ is called the *antipode* of $\mathcal{H}$ and required to satisfy the following equalities:

$$\mathscr{S}^2 = \mathrm{Id}, \qquad \mathsf{t}(\varepsilon(u)) = \mathscr{S}(u_{(1)})u_{(2)}, \qquad \mathsf{s}(\varepsilon(u)) = u_{(1)}\mathscr{S}(u_{(2)}), \quad \text{for every } u \in \mathcal{H}, \tag{13}$$

where we used Sweedler's notation for the comultiplication, summation is understood. The algebras A and $\mathcal{H}$ are called, respectively, *the base algebra* and *the total algebra* of the Hopf algebroid $(A, \mathcal{H})$.

As commutative Hopf algebra leads to an affine group scheme, a Hopf algebroid leads to an affine groupoid scheme (i.e. a presheaf of groupoids). Precisely, given a Hopf algebroid $(A, \mathcal{H})$ and an algebra C, reversing the structure of $(A, \mathcal{H})$ we have, in a natural way, a groupoid structure

$$\mathscr{H}(C): \ \mathcal{H}(C) \underset{\mathsf{t}^*}{\overset{\mathsf{s}^*}{\rightrightarrows}} \!\!\xleftarrow{\varepsilon^*}\! A(C). \tag{14}$$

This structure is explicitly given as follows: the source and the target of a given arrow $g \in \mathcal{H}(C)$ are, respectively, $\mathsf{s}^*(g) = g \circ \mathsf{s}$ and $\mathsf{t}^*(g) = g \circ \mathsf{t}$; the inverse is $g^{-1} = g \circ \mathscr{S}$. Given another arrow $f \in \mathcal{H}(C)$ with $\mathsf{t}^*(f) = \mathsf{s}^*(g)$, then the multiplication is defined by the following algebra map

$$gf : \mathcal{H} \longrightarrow C, \quad \Big(u \longmapsto f(u_{(1)})g(u_{(2)})\Big),$$

summation always understood. The identity arrow of an object $x \in A(C)$ is $\varepsilon^*(x) = x \circ \varepsilon$.

The functor $\mathscr{H}$ is refereed to as *the associated presheaf of groupoids* of the Hopf algebroid $(A, \mathcal{H})$, and the groupoids of equation (14) are called *the fibres of* $\mathscr{H}$. Depending on the handled situation, we will employ different notations for the fibres of $\mathscr{H}$ at an algebra C:

$$\mathscr{H}(C) = (\mathcal{H}(C), A(C)) = (\mathscr{H}_1(C), \mathscr{H}_0(C)).$$

The presheaf of groupoids $\mathscr{H}^{op}$ is defined to be the presheaf whose fibre at C is the opposite groupoid $\mathscr{H}(C)^{op}$ (i.e. the same groupoid with the source interchanged by the target).

Examples of Hopf algebroids can be then provided using well known constructions in abstract groupoids, as we have seen in subsection 2.1.

Example 3.1. Given an algebra A and set $\mathcal{H} := A \otimes A$. Then the pair $(A, \mathcal{H})$ is clearly a Hopf algebroid with structure $\mathsf{s} : A \to \mathcal{H}$, $a \mapsto a \otimes 1$, $\mathsf{t} : A \to \mathcal{H}$, $a \mapsto 1 \otimes a$; $\Delta(a \otimes a') = (a \otimes 1) \otimes_A (1 \otimes a')$, $\varepsilon(a \otimes a') = aa'$, $\mathscr{S}(a \otimes a') = a' \otimes a$. Clearly, the fibres of the associated presheaf of groupoids $\mathscr{H}$ are groupoids of pairs, as in Example 2.2. Thus, $\mathscr{H} \cong (\mathscr{A} \times \mathscr{A}, \mathscr{A})$ as presheaves of groupoids, where $\mathscr{A}$ is the presheaf of sets attached to the algebra A.

Example 3.2. Let $(B, \Delta, \varepsilon, \mathcal{S})$ be a commutative Hopf $\Bbbk$-algebra and A a commutative right B-comodule algebra with coaction $A \to A \otimes B$, $a \mapsto a_{(0)} \otimes a_{(1)}$. This means that A is right B-comodule and the coaction is an algebra map, see [22, §, 4]. Let $\mathscr{B}$ be the affine $\Bbbk$-group attached to B. To any algebra C, one associated in a natural way, the following action groupoid as in Example 2.3:

$$(\mathscr{A} \times \mathscr{B})(C): \ A(C) \times B(C) \rightrightarrows A(C),$$

where the source is given by the action $(x, g) \mapsto xg$ sending $a \mapsto (xg)(a) = x(a_{(0)})g(a_{(1)})$, and the target is the first projection. Consider, on the other hand, the algebra $\mathcal{H} = A \otimes B$ with algebra extension $\eta : A \otimes A \to \mathcal{H}$, $a' \otimes a \mapsto a'a_{(0)} \otimes a_{(1)}$. Then $(A, \mathcal{H})$ has a structure of Hopf algebroids, known as a *split Hopf algebroid*:

$$\Delta(a \otimes b) = (a \otimes b_{(1)}) \otimes_A (1_A \otimes b_{(2)}), \quad \varepsilon(a \otimes b) = a\varepsilon(b), \quad \mathscr{S}(a \otimes b) = a_{(0)} \otimes a_{(1)}\mathcal{S}(b).$$

Obviously, the associated presheaf of groupoids $\mathscr{H}^{op}$ (where $\mathscr{H}$ is the one associated to $(A, \mathcal{H})$) is canonically isomorphic to the action groupoids $\mathscr{A} \times \mathscr{B}$. Thus, we have an isomorphism $\mathscr{H}^{op} \cong (\mathscr{A} \times \mathscr{B}, \mathscr{A})$ of presheaves of groupoids.

Example 3.3. Let A be an algebra and consider the commutative polynomial Laurent ring over $A \otimes A$, that is, $\mathcal{H} = (A \otimes A)[X, X^{-1}]$ with the canonical injection $\eta : A \otimes A \to \mathcal{H}$. The pair $(A, \mathcal{H})$ is a Hopf algebroid with structure maps

$$\Delta((a \otimes a')X) = ((a \otimes_A 1)X) \otimes_A ((1 \otimes a')X), \quad \varepsilon((a \otimes a')X) = aa', \quad \mathscr{S}((a \otimes a')X) = (a' \otimes a)X^{-1}.$$

The fibres of the associated presheaf $\mathscr{H}$ are described using the induced groupoid by the multiplicative affine $\Bbbk$-group, in the sense of Example 2.4. Precisely, take a ring C, then $\mathcal{H}(C)$ is canonically bijective

to the set $A(C) \times \mathcal{G}_m(C) \times A(C)$, where $\mathcal{G}_m$ is the multiplicative affine $\Bbbk$-group. This in fact induces, in a natural way, an isomorphisms of groupoids $(\mathcal{H}(C), A(C)) \cong (A(C) \times \mathcal{G}_m(C) \times A(C), A(C))$, where the later is the induced groupoid by the group $\mathcal{G}_m(C)$, see Example 2.4. As presheaves of groupoids, we have then an isomorphism $\mathscr{H} \cong (\mathscr{A} \times \mathcal{G}_m \times \mathscr{A}, \mathscr{A})$, where as before $\mathscr{A}$ is the presheaf attached to the algebra A.

There is in fact a more general construction: Take any commutative Hopf $\Bbbk$-algebra B, then for any algebra A, the pair $(A, A \otimes B \otimes A)$ admits a canonical structure of Hopf algebroid whose associated presheaf is also of the form $(\mathscr{A} \times \mathscr{B} \times \mathscr{A}, \mathscr{A})$, where $\mathscr{A}$ and $\mathscr{B}$ are as before.

The notion of characters group in commutative Hopf algebras context, is naturally extended to that of characters groupoids in commutative Hopf algebroids:

Definition 3.4. Let $(A, \mathcal{H})$ be a Hopf algebroid over a field $\Bbbk$ and $\mathscr{H}$ its associated presheaf of groupoids. The *characters groupoid* of $(A, \mathcal{H})$ is the fibre groupoid $\mathscr{H}(\Bbbk)$ at the base filed $\Bbbk$.

Since this groupoid might be empty (i.e. could be a category without objects), to avoid this extreme situation, *we will always assume that* $A \neq 0$ *and* $A(\Bbbk) \neq \emptyset$, *for any handled Hopf algebroid* $(A, \mathcal{H})$ *over* $\Bbbk$.

The following definition, which we will frequently use in the sequel, can be found in [9].

Definition 3.5. Let $(A, \mathcal{H})$ be a Hopf algebroid and $\mathscr{H}$ its associated presheaf of groupoids. Given an algebra C, consider the fibre groupoid $\mathscr{H}(C)$ (notice that $A(C) \neq \emptyset$ since $A(\Bbbk)$ is assumed so). Two objects $x, y \in A(C)$ are said to be *locally isomorphic* (in the sense of the fpqc topology) if there exists a faithfully flat extension $p : C \to C'$ and an arrow $g \in \mathcal{H}(C')$ such that

$$p \circ x = g \circ \mathsf{s}, \quad \text{ and } \quad p \circ y = g \circ \mathsf{t}.$$

We also say that any two objects of $\mathscr{H}$ (without specifying the algebra C) are *locally isomorphic*.

In case that the presheaf $\mathscr{H}$ is *transitive*, that is, each of its fibres $\mathscr{H}(C)$ is a transitive groupoid, then obviously any two objects of $\mathscr{H}$ are locally isomorphic. For instance, this is the case for the Hopf algebroid $(A, A \otimes A)$, since in this case each of the groupoid's $\mathscr{H}(C)$ is the groupoid of pairs, see Example 2.2. The same holds true for the class of Hopf algebroids described in Example 3.3.

A *morphism* $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ *of Hopf algebroids* consists of a pair $\boldsymbol{\phi} = (\phi_0, \phi_1)$ of algebra maps $\phi_0 : A \to B$ and $\phi_1 : \mathcal{H} \to \mathcal{K}$ that are compatible, in a canonical way, with the structure maps of both $\mathcal{H}$ and $\mathcal{K}$. That is, the equalities

$$\phi_1 \circ \mathsf{s} \;=\; \mathsf{s} \circ \phi_0, \qquad \phi_1 \circ \mathsf{t} \;=\; \mathsf{t} \circ \phi_0, \tag{15}$$

$$\Delta \circ \phi_1 \;=\; \chi \circ (\phi_1 \otimes_A \phi_1) \circ \Delta, \qquad \varepsilon \circ \phi_1 \;=\; \phi_0 \circ \varepsilon, \tag{16}$$

$$\mathscr{S} \circ \phi_1 \;=\; \phi_1 \circ \mathscr{S}, \tag{17}$$

hold, where χ is the obvious map $\chi : \mathcal{K} \otimes_A \mathcal{K} \to \mathcal{K} \otimes_B \mathcal{K}$, and where no distinction between the structure maps of $\mathcal{H}$, $\mathcal{K}$ was made. Clearly, any morphism $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ of Hopf algebroids induces (in the opposite way) a morphism between the associated presheaves of groupoids, which is given over each fibre by

$$({\phi_0}^*, {\phi_1}^*) : \mathscr{K}(C) = (\mathcal{K}(C), B(C)) \longrightarrow \mathscr{H}(C) = (\mathcal{H}(C), A(C)), \quad \text{sending } (g, x) \mapsto (g \circ \phi_1, x \circ \phi_0).$$

In this way, the construction of the following example corresponds to the construction of the induced groupoid as expounded in Example 2.4.

Example 3.6. Given a Hopf algebroid $(A, \mathcal{H})$ and an algebra map $\phi : A \to B$, then the pair of algebras

$$(B, \mathcal{H}_\phi) := (B, B \otimes_A \mathcal{H} \otimes_A B)$$

is a Hopf algebroid known as *the base change Hopf algebroid* of $(A, \mathcal{H})$, and $(\phi, \phi_1) : (A, \mathcal{H}) \to (B, \mathcal{H}_\phi)$ is a morphism of Hopf algebroids, where $\phi_1 : \mathcal{H} \to \mathcal{H}_\phi$ sends $u \mapsto 1_B \otimes_A u \otimes_A 1_B$. Moreover, as in the case of groupoids, see subsection 2.1, any morphism $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ factors through *the base change morphism* $(A, \mathcal{H}) \to (B, \mathcal{H}_{\phi_0})$, by using the map $\mathcal{H}_{\phi_0} \to \mathcal{K}$, $b \otimes_A u \otimes_A b' \mapsto bb'\phi_1(u)$.

The aforementioned relation with the induced groupoids comes out as follows. Take an algebra C and consider the associated groupoid $\mathscr{H}(C)$. Then the map $\phi^* : B(C) \to A(C)$ leads, as in Example 2.4, to the induced groupoid $\mathscr{H}(C)^{\phi(C)}$. This in fact determines a presheaf of groupoids $C \to \mathscr{H}(C)^{\phi^*}$ which can be easily shown to be represented by the pair of algebras $(B, \mathcal{H}_\phi)$.

We finish this subsection by recalling the construction of the 2-category of flat Hopf algebroids. A Hopf algebroid $(A, \mathcal{H})$ is said to be *flat*, when ${}_{\mathsf{s}}\mathcal{H}$ (or $\mathcal{H}_{\mathsf{t}}$) is a flat A-module. Notice, that in this case s as well as t are faithfully flat extensions. As was mentioned before, groupoids, functors, and natural transformations form a 2-category. Analogously (flat) Hopf algebroids over the ground field $\Bbbk$ form a 2-category, as was observed in [23, §3.1]. Precisely, 0-cells are Hopf algebroids, or even flat ones, 1-cells are morphisms of Hopf algebroids, and for two 1-cells $(\phi_0, \phi_1), (\psi_0, \psi_1) : (A, \mathcal{H}) \to (B, \mathcal{K})$, a 2-cell $\mathfrak{c} : (\phi_0, \phi_1) \to (\psi_0, \psi_1)$ is defined to be an algebra map $\mathfrak{c} : \mathcal{H} \to B$ that makes the diagrams

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\mathfrak{c}"] & B \\ A \arrow[u, "\mathsf{s}"] \arrow[ur, "\phi_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r, "\mathfrak{c}"] & B \\ A \arrow[u, "\mathsf{t}"] \arrow[ur, "\psi_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & \mathcal{H} \otimes_A \mathcal{H} \arrow[d, "m_{\mathcal{K}} \circ (\phi_1 \otimes_A (\mathsf{t} \circ \mathfrak{c}))"] \\ \mathcal{H} \otimes_A \mathcal{H} \arrow[rr, "m_{\mathcal{K}} \circ ((\mathsf{s} \circ \mathfrak{c}) \otimes_A \psi_1)"'] & & \mathcal{K} \end{tikzcd} \tag{18}$$

commutative, where $m_{\mathcal{K}}$ denotes the multiplication of $\mathcal{K}$. The identity 2-cell for (ϕ_0, ϕ_1) is given by $1_\phi := \phi_0 \circ \varepsilon$. The tensor product (or vertical composition) of 2-cells is given as

$$\mathfrak{c}' \circ \mathfrak{c} : (\phi_0, \phi_1) \xrightarrow{\ \mathfrak{c}\ } (\psi_0, \psi_1) \xrightarrow{\ \mathfrak{c}'\ } (\xi_0, \xi_1),$$

where

$$\mathfrak{c}' \circ \mathfrak{c} : \mathcal{H} \to B, \quad u \mapsto \mathfrak{c}(u_{(1)})\mathfrak{c}'(u_{(2)}). \tag{19}$$

3.3. **Comodules, bicomodules (algebras), and presheaves of orbit sets.** A *right $\mathcal{H}$-comodule* is a pair (M, ϱ) consisting of right A-module M and right A-linear map (referred to as the *coaction*) $\varrho : M \to M \otimes_A {}_{\mathsf{s}}\mathcal{H}$, $m \mapsto m_{(0)} \otimes_A m_{(1)}$ (summation understood) satisfying the usual counitary and coassociativity properties. Morphisms between right $\mathcal{H}$-comodules (or right *$\mathcal{H}$-colinear map*) are A-linear maps compatible with both coactions. The *category of all right $\mathcal{H}$-comodules* is denoted by $\mathsf{Comod}_{\mathcal{H}}$. This is a symmetric monoidal $\Bbbk$-linear category with identity object A endowed with the coaction $\mathsf{t} : A \to \mathcal{H} \cong A \otimes_A {}_{\mathsf{s}}\mathcal{H}$.

The tensor product in $\mathsf{Comod}_{\mathcal{H}}$ is defined via the so called *the diagonal coaction*. Precisely, given (M, ϱ) and (N, ϱ) two (right) $\mathcal{H}$-comodules. Then the tensor product $M \otimes_A N$ is endowed with the following (right) $\mathcal{H}$-coaction:

$$\varrho_{M \otimes_A N} : M \otimes_A N \longrightarrow (M \otimes_A N) \otimes_A \mathcal{H}, \quad \Big(m \otimes_A n \longmapsto (m_{(0)} \otimes_A n_{(0)}) \otimes_A m_{(1)} n_{(1)}\Big). \tag{20}$$

The vector space of all $\mathcal{H}$-colinear maps between two comodules (M, ϱ) and (N, ϱ) will be denoted by $\mathrm{Hom}^{\mathcal{H}}(M, N)$, and the endomorphism ring by $\mathrm{End}^{\mathcal{H}}(M)$.

Inductive limit and cokernels do exist in $\mathsf{Comod}_{\mathcal{H}}$, and can be computed in A-modules. Furthermore, it is well known that the underlying module ${}_{\mathsf{s}}\mathcal{H}$ is flat if and only if $\mathsf{Comod}_{\mathcal{H}}$ is a Grothendieck category and the forgetful functor $\mathscr{U}_{\mathcal{H}} : \mathsf{Comod}_{\mathcal{H}} \to \mathsf{Mod}_A$ is exact. As it can be easily checked, the forgetful functor $\mathscr{U}_{\mathcal{H}}$ has a right adjoint functor $- \otimes_A {}_{\mathsf{s}}\mathcal{H} : \mathsf{Mod}_A \to \mathsf{Comod}_{\mathcal{H}}$.

The full subcategory of right $\mathcal{H}$-comodules whose underlying A-modules are finitely generated is denoted by $\mathsf{comod}_{\mathcal{H}}$. The category of left $\mathcal{H}$-comodules is analogously defined, and it is isomorphic via the antipode map to the category of right $\mathcal{H}$-comodules.

A (right) $\mathcal{H}$-comodule algebra can be defined as a commutative monoid in the symmetric monoidal category $\mathsf{Comod}_{\mathcal{H}}$. This is a commutative algebra extension $\sigma : A \to R$ where the associated A-module R_σ is also a (right) $\mathcal{H}$-comodule whose coaction $\varrho_R : R_\sigma \to R_\sigma \otimes_A {}_{\mathsf{s}}\mathcal{H}$ is an algebra map, which means that

$$\varrho_R(1_R) \;=\; 1_R \otimes_A 1_{\mathcal{H}}, \quad \varrho_R(rr') \;=\; r_{(0)} r'_{(0)} \otimes_A r_{(1)} r'_{(1)}, \quad \text{for every } r, r' \in R.$$

This of course induces a (right) $\mathscr{H}$-action on the presheaf of sets $\mathscr{R}$ associated to R. Precisely, given an algebra C, consider the map $\sigma^* : R(C) \to A(C)$ sending $x \mapsto x \circ \sigma$, and set

$$R(C)\, {}_{\sigma^*}\!\times_{\mathsf{s}^*}\, \mathscr{H}_1(C) \to R(C), (x, g) \mapsto xg, \text{ where } xg : R \to C,\ r \mapsto x(r_{(0)})\, g(r_{(1)}). \tag{21}$$

Then this defines, in a natural way, a right action of the groupoid $\mathscr{H}(C)$ on the set $R(C)$, in the sense of Definition 2.6. Equivalently such an action can be expressed by a pair morphism of presheaves $\mathscr{R} \to \mathscr{H}_0$ and $\mathscr{R}\, {}_{\sigma^*}\!\times_{\mathsf{s}^*}\, \mathscr{H}_1 \to \mathscr{R}$ satisfying pertinent compatibilities. In this way, an action of a presheaf of groupoids on a presheaf of sets, can be seen as a natural generalization to the groupoids framework of the notion of an action of group scheme on a scheme [10, n°3, page 160], or more formally, as a generalization of the notion of "objet à groupe d'opérateurs à droite" [14, Chapitre III, §1.1].

For a right $\mathcal{H}$-comodule algebra (R, σ), we have the *subalgebra of coinvariants* defined by

$$R^{coinv_{\mathcal{H}}} := \Big\{ r \in R | \, \varrho_R(r) = r \otimes_A 1_{\mathcal{H}} \Big\}.$$

Denote by $\mathscr{R}^{\mathscr{H}}$ the presheaf of sets represented by the algebra $R^{coinv_{\mathcal{H}}}$. On the other hand, we have the presheaf defining the orbit sets, which is given as follows: Take an algebra C and consider the action (21), we then obtain the orbits set $R(C)/\mathscr{H}(C)$. Clearly this establishes a functor: $C \longrightarrow R(C)/\mathscr{H}(C)$ yielding to a presheaf $\mathscr{O}_{\mathscr{H}}(\mathscr{R})$ with a canonical morphism of presheaves $\mathscr{O}_{\mathscr{H}}(\mathscr{R}) \longrightarrow \mathscr{R}^{\mathscr{H}}$.

Remark 3.7. An important example of the previous construction is the case of the right $\mathcal{H}$-comodule algebra (A, t). In this case we have a commutative diagram of presheaves:

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "\tau"] \arrow[dr, "\zeta"'] & & \mathscr{A}^{\mathscr{H}} \\ & \mathscr{O}_{\mathscr{H}}(\mathscr{A}) \arrow[ur] & \end{tikzcd}$$

where as before $\mathscr{A}$ is the presheaf represented by the algebra A and $\mathscr{A}^{\mathscr{H}}$ is represented by $A^{coinv_{\mathcal{H}}}$.

Notice that the presheaf $\mathscr{O}_{\mathscr{H}}(\mathscr{A})$ is not necessarily represented by $A^{coinv_{\mathcal{H}}}$, thus, the right hand map in the previous diagram is not in general an isomorphism of presheaves, see [24, page 54].

In this direction, both $\mathscr{A} \, {}_{\tau}\!\times_{\tau} \, \mathscr{A}$ and $\mathscr{A} \, {}_{\zeta}\!\times_{\zeta} \, \mathscr{A}$ enjoy a structure of presheaves of groupoids with fibres are the groupoids described in Examples 2.2(2). Nevertheless, $\mathscr{A} \, {}_{\zeta}\!\times_{\zeta} \, \mathscr{A}$ is not necessarily representable. Furthermore, there is a commutative diagram

$$\begin{tikzcd} \mathscr{H} \arrow[rr] \arrow[dr] & & \mathscr{A} \, {}_{\tau}\!\times_{\tau} \, \mathscr{A} \\ & \mathscr{A} \, {}_{\zeta}\!\times_{\zeta} \, \mathscr{A} \arrow[ur] & \end{tikzcd}$$

of presheaves of groupoids.

Given two Hopf algebroids $(A, \mathcal{H})$ and $(B, \mathcal{K})$, *the category of* $(\mathcal{H}, \mathcal{K})$*-bicomodules* is defined as follows. An object in this category is a triple (M, λ, ϱ) consisting of left $\mathcal{H}$-comodule (M, λ) and right $\mathcal{K}$-comodule (M, ϱ) such that λ is a morphism of right $\mathcal{K}$-comodules, or equivalently ϱ is a morphism of left $\mathcal{H}$-comodules. Morphisms between bicomodules are simultaneously left and right comodules morphisms. On the other hand, the pair of tensor product $(A \otimes B, \mathcal{H}^o \otimes \mathcal{K})$ admits, in a canonical way, a structure of Hopf algebroid, where $(A, \mathcal{H}^o)$ is the opposite Hopf algebroid (i.e. the source and the target are interchanged, or equivalently, the fibres of the associated presheaf are the opposite groupoids $\mathscr{H}(C)^{op}$). This is *the tensor product Hopf algebroid*, and its category of right comodules is canonically identified with the category of $(\mathcal{H}, \mathcal{K})$-bicomodules. Thus bicomodules form also a symmetric monoidal category.

A *bicomodule algebra* is a bicomodule which is simultaneously a left comodule algebra and right comodule algebra. As above, by using the actions of equation (21) a bicomodule algebra leads to a presheaf of groupoid bisets. That is, a presheaf with fibres abstract groupoid-bisets, in the sense of Definition 2.7.

3.4. Weak equivalences and principal bundles between Hopf algebroids.

Any morphism $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ of Hopf algebroids induces a symmetric monoidal functor

$$\boldsymbol{\phi}_* := \mathscr{U}_{\mathcal{H}}(-) \otimes_A B : \mathsf{Comod}_{\mathcal{H}} \longrightarrow \mathsf{Comod}_{\mathcal{K}},$$

where, for any $\mathcal{H}$-comodule (M, ϱ), the $\mathcal{K}$-comodule structure of $M \otimes_A B$ is given by

$$M \otimes_A B \longrightarrow (M \otimes_A B) \otimes_B \mathcal{K}, \quad m \otimes_A b \longmapsto (m_{(0)} \otimes_A 1_B) \otimes_B \phi_1(m_{(1)})\mathsf{t}(b).$$

Following [16, Definition 6.1], $\boldsymbol{\phi}$ is said to be a *weak equivalence* whenever $\boldsymbol{\phi}_*$ is an equivalence of categories. In this case, $\mathsf{Comod}_{\mathcal{H}}$ and $\mathsf{Comod}_{\mathcal{K}}$ are equivalent as symmetric monoidal categories.

Notice that if $\boldsymbol{\phi}$ is a weak equivalence, then so is the associated morphism between the tensor product Hopf algebroids $\boldsymbol{\phi}^o \otimes \boldsymbol{\phi} : (A \otimes A, \mathcal{H}^o \otimes \mathcal{H}) \to (B \otimes B, \mathcal{K}^o \otimes \mathcal{K})$, which induces then a symmetric monoidal equivalence between the categories of $\mathcal{H}$-bicomodules and $\mathcal{K}$-bicomodules.

Two Hopf algebroids $(A, \mathcal{H})$ and $(B, \mathcal{K})$ are said to be *weakly equivalent* if there exists a diagram

$$\begin{tikzcd} & (C, \mathcal{J}) & \\ (A, \mathcal{H}) \arrow[ur] & & (B, \mathcal{K}), \arrow[ul] \end{tikzcd}$$

of weak equivalences.

As was shown in [11] weak equivalences between flat Hopf algebroids are strongly related to principal bi-bundles. Such a relation is in part a consequence of the analogue one for abstract groupoids as was shown in Proposition 2.13 (see Remark 2.14).

Recall that, for two flat Hopf algebroids $(A, \mathcal{H})$ and $(B, \mathcal{K})$, a *left principal* $(\mathcal{H}, \mathcal{K})$*-bundle* is a three-tuple (P, α, β) which consist of diagram of commutative algebras $\alpha : A \to P \leftarrow B : \beta$ where the (A, B)-bimodule ${}_{\alpha}P_{\beta}$ enjoys a structure of an $(\mathcal{H}, \mathcal{K})$-bicomodule algebra such that

(PB1) $\beta : B \to P$ is a faithfully flat extension (the local triviality of the bundle in the fpqc topology);
(PB2) the canonical map $\mathsf{can}_{P,\mathcal{H}} : P \otimes_B P \to \mathcal{H} \otimes_A P$ sending $p \otimes_B p' \mapsto p_{(0)} \otimes_A p_{(1)} p'$ is bijective.

This is a natural generalization of the notion of *Torsor*, where the group object is replaced by groupoid object, see [14, Définition 1.4.1, page 117] and also [10, Chapter III, §4].

Right principal bundles and bi-bundles are clearly understood. For instance to each left principal bundle (P, α, β), one can define a right principal bundle on the opposite bicomodule P^{co}. As in the case of groupoids, see subsection 2.3, a simpler example of left principal bundle is *the unit bundle* $\mathscr{U}(\mathcal{H})$ which is $\mathcal{H}$ with its structure of $\mathcal{H}$-bicomodule algebra. A *trivial bundle* attached to a given morphism of Hopf algebroids $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$, is the one whose underlying bicomodule algebra is of the form $P := \mathcal{H} \otimes_A B$, that is, the pull-back bundle $\boldsymbol{\phi}^*(\mathscr{U}(\mathcal{H}))$ of the unit bundle $\mathscr{U}(\mathcal{H})$.

Parallel to subsection 2.3, for any bicomodule algebra, and thus for any left principal bundle, one can associate the so called *the two-sided translation Hopf algebroid*, which is denoted by $(P, \mathcal{H} \ltimes P \rtimes \mathcal{K})$. The underlying pair of algebras is $(P, \mathcal{H}_{\mathsf{s}} \otimes_A P \otimes_{B\,\mathsf{s}} \mathcal{K})$ and its structure of Hopf algebroid is given as follows:

- the source and target are given by

$$\mathsf{s}(p) := 1_{\mathcal{H}} \otimes_A p \otimes_B 1_{\mathcal{K}}, \quad \mathsf{t}(p) := \mathscr{S}(p_{(-1)}) \otimes_A p_{(0)} \otimes_B p_{(1)};$$

- the comultiplication and counit are given by:

$$\Delta(u \otimes_A p \otimes_B w) := (u_{(1)} \otimes_A p \otimes_B w_{(1)}) \otimes_P (u_{(2)} \otimes_A 1_P \otimes_B w_{(2)}), \ \ \varepsilon(u \otimes_A p \otimes_B w) := \alpha(\varepsilon(u)) p \beta(\varepsilon(w));$$

- whereas the antipode is defined as:

$$\mathscr{S}(u \otimes_A p \otimes_B w) := \mathscr{S}(u p_{(-1)}) \otimes_A p_{(0)} \otimes_B p_{(1)} \mathscr{S}(w).$$

Furthermore, there is a diagram

$$\begin{array}{ccccc} & & (P, \mathcal{H} \ltimes P \rtimes \mathcal{K}) & & \\ & \nearrow^{\boldsymbol{\alpha}=(\alpha,\alpha_1)} & & \nwarrow^{\boldsymbol{\beta}=(\beta,\beta_1)} & \\ (A, \mathcal{H}) & & & & (B, \mathcal{K}) \end{array} \tag{22}$$

of Hopf algebroids, where α_1 and β_1 are, respectively, the maps $u \mapsto u \otimes_A 1_P \otimes_B 1_{\mathcal{K}}$ and $w \mapsto 1_{\mathcal{H}} \otimes_A 1_P \otimes_B w$. It is easily check that $(P, \mathcal{H} \ltimes P \rtimes \mathcal{K})$ is a flat Hopf algebroid whenever $(A, \mathcal{H})$ and $(B, \mathcal{K})$ they are so.

Remark 3.8. Let us make some general observations and remarks on the above constructions.

It is noteworthy to mention that the fibres of the presheaf associated to a left principal bundle are not necessarily principal bisets over the fibres groupoids, in the sense of Definition 2.8 (but possibly the entry presheaf is locally so in the fpqc topology sense). To be precise, let $\mathscr{P}$ denote as before the presheaf of sets associated to the algebra P. This is a presheaf of $(\mathscr{H}, \mathscr{K})$-bisets, that is, using left and right actions of equation (21), for any algebra C, we have that the fibre $\mathscr{P}(C)$ is actually an $(\mathscr{H}(C), \mathscr{K}(C))$-biset as in Definition 2.7. However, $\mathscr{P}(C)$ is not necessarily a left principal biset. Nevertheless, it is easily seen that the associated presheaf of two-sided translation groupoids is represented by the two-sided translation Hopf algebroid $(P, \mathcal{H} \ltimes P \rtimes \mathcal{K})$.

On the other hand, notice that even if $(A, \mathcal{H})$ and $(B, \mathcal{K})$ are Hopf algebras, that is, Hopf algebroids with source equal the target, then the two-sided translation Hopf algebroid is not necessary a Hopf algebra. Thus, new Hopf algebroids can be constructed using bicomodules algebras over Hopf algebras.

Lastly, as in [11], two weakly equivalent flat Hopf algebroids are shown to be connected by a principal bibundle, for which the diagram (22) becomes a diagram of weak equivalences, see *op. cit.* for more characterizations of weak equivalences.

3.5. **Dualizable right comodules.** Recall that a (right) $\mathcal{H}$-comodule (M, ϱ) is said to be *dualizable*, if there is another (right) $\mathcal{H}$-comodule (N, ϱ_N) and two morphisms of comodules

$$\mathsf{ev} : (N \otimes_A M, \varrho_{N \otimes_A M}) \to (A, \mathsf{t}), \text{ and } \mathsf{db} : (A, \mathsf{t}) \to (M \otimes_A N, \varrho_{M \otimes_A N}) \tag{23}$$

satisfying, up to natural isomorphisms, the usual triangle properties. Taking the underlying A-linear maps $(\mathsf{ev}, \mathsf{db})$ and using these triangle properties, one shows that $\mathscr{U}_{\mathcal{H}}(N) \cong M^* = \mathrm{Hom}_A(M, A)$. Thus, the underlying A-module of any dualizable comodule is finitely generated and projective. Moreover, the dual in the category $\mathsf{Comod}_{\mathcal{H}}$ is, up to an isomorphism, (M^*, ϱ_{M^*}) with the following coaction:

$$\varrho_{M^*} : M^* \to M^* \otimes_A \mathcal{H}, \quad \Big(\varphi \longmapsto e_i^* \otimes_A \mathsf{t}(\varphi(e_{i,(0)}))\mathscr{S}(e_{i,(1)})\Big), \tag{24}$$

where $\{e_i, e_i^*\}$ is a dual basis for M_A, that is, $\mathsf{db}(1_A) = \sum_i e_i \otimes_A e_i^*$. The converse holds true as well, that is, dualizable objects in $\mathsf{Comod}_{\mathcal{H}}$ are, up to natural isomorphisms, precisely the objects of the subcategory $\mathsf{comod}_{\mathcal{H}}$ which are projective as A-modules. This fact will be implicitly used below, so for sake of completeness, we give here an elementary detailed proof.

LEMMA 3.9. *Let (M, ϱ) be a right $\mathcal{H}$-comodule whose underlying A-module M is finitely generated and projective, and consider (M^*, ϱ_{M^*}) as right $\mathcal{H}$-comodule with the coaction given by* (24). *Then (M, ϱ) is a dualizable object in $\mathsf{Comod}_{\mathcal{H}}$ with dual object (M^*, ϱ_{M^*}). In particular, the full subcategory of dualizable right $\mathcal{H}$-comodules consists of those comodules with finitely generated and projective underlying A-modules.*

Proof. The proof uses the following A-linear map:

$$\zeta : M^* \longrightarrow \mathrm{Hom}_A(M, \mathcal{H}_{\mathsf{t}}), \quad \Big(\varphi \longmapsto [m \mapsto \mathsf{s}(\varphi(m_{(0)})m_{(1)}]\Big),$$

where $\mathrm{Hom}_A(M, \mathcal{H}_{\mathsf{t}})$ is an A-module via the source map on $\mathcal{H}$. For a fixed $\varphi \in M^*$, it is clear that $\zeta(\varphi) : M \to \mathcal{H}$ is right $\mathcal{H}$-colinear. Let $\{e_i^*, e_i\}_i$ be a dual basis for the A-module M. So we need to check that the following two maps

$$\mathsf{db} : A \longrightarrow M \otimes_A M^*, \quad \Big(1_A \longmapsto \sum_i e_i \otimes_A e_i^*\Big), \qquad \mathsf{ev} : M^* \otimes_A M \longrightarrow A, \quad \Big(\varphi \otimes_A m \longmapsto \varphi(m)\Big),$$

are morphisms of right $\mathcal{H}$-comodules. Let us check first that ev is right $\mathcal{H}$-colinear:

$$\begin{aligned}
(\mathsf{ev} \otimes_A \mathcal{H}) \circ \varrho_{M^* \otimes_A M}(\varphi \otimes_A m) &= \sum_j \mathsf{s}(e_j^*(m_{(0)}))\mathsf{t}(\varphi(e_{j,(0)}))\mathscr{S}(e_{j,(1)})m_{(1)} \\
&= \sum_j \mathsf{s}(e_j^*(m_{(0)}))\mathscr{S}\Big(\mathsf{s}(\varphi(e_{j,(0)}))\, e_{j,(1)}\Big)m_{(1)} \\
&= \sum_j \mathscr{S}\Big(\mathsf{t}(e_j^*(m_{(0)}))\zeta(\varphi)(e_j))\Big)m_{(1)} \\
&= \mathscr{S}\Big(\zeta(\varphi)(m_{(0)})\Big)m_{(1)} \;=\; \mathsf{t}(\varphi(m_{(0)}))\mathscr{S}(m_{(1)})m_{(2)} \\
&= \mathsf{t}(\varphi(m_{(0)}))\mathsf{t}(\varepsilon(m_{(1)})) \;=\; \mathsf{t}(\varphi(m)) \;=\; \varrho_A \circ \mathsf{ev}(\varphi \otimes_A m).
\end{aligned}$$

As for the map db, we first use the coassociativity of ϱ_{M^*}, to derive the following equality:

$$\sum_i \zeta(\varphi)(e_{i,(0)}) \otimes_A e_{i,(1)} \otimes_A e_i^* \;=\; \sum_{i,j} \zeta(\varphi)(e_i) \otimes_A \zeta(e_i^*)(e_j) \otimes_A e_j^* \;\in \mathcal{H} \otimes_A \mathcal{H} \otimes_A M^*, \tag{25}$$

for every $\varphi \in M^*$. Applying the map $(m_{\mathcal{H}} \otimes_A M^*) \circ (\mathcal{H} \otimes_A \mathscr{S} \otimes_A M^*)$ to equality (25), where $m_{\mathcal{H}}$ is the multiplication of $\mathcal{H}$, we obtain the following one

$$\begin{aligned}
\sum_i \zeta(\varphi)(e_{i,(0)})\mathscr{S}(e_{i,(1)}) \otimes_A e_i^* &= \sum_{i,j} \zeta(\varphi)(e_i)\mathscr{S}\Big(\zeta(e_i^*)(e_j)\Big) \otimes_A e_j^* \\
\sum_i \mathsf{s}(\varphi(e_{i,(0)}))e_{i,(1)}\mathscr{S}(e_{i,(2)}) \otimes_A e_i^* &= \sum_j \Big(\sum_i \zeta(\varphi)(e_i)\mathscr{S}\Big(\zeta(e_i^*)(e_j)\Big)\Big) \otimes_A e_j^* \\
\sum_i \mathsf{s}(\varphi(e_{i,(0)}))\mathsf{s}(\varepsilon(e_{i,(1)})) \otimes_A e_i^* &= \sum_j \Big(\sum_i \zeta(\varphi)(e_i)\mathscr{S}\Big(\zeta(e_i^*)(e_j)\Big)\Big) \otimes_A e_j^*
\end{aligned}$$

$$\sum_i \mathsf{s}(\varphi(e_i))1_{\mathcal{H}} \otimes_A e_i^* \;=\; \sum_j \Big(\sum_i \zeta(\varphi)(e_i)\mathscr{S}\big(\zeta(e_i^*)(e_j)\big)\Big) \otimes_A e_j^*,$$

which by the dual basis property means that:

$$\mathsf{s}(\varphi(m))1_{\mathcal{H}} \;=\; \sum_i \zeta(\varphi)(e_i)\mathscr{S}\big(\zeta(e_i^*)(m)\big), \tag{26}$$

for every $\varphi \in M^*$ and $m \in M$. Now, using equation (26) in conjunction with the dual basis properties, we show that

$$\varrho_{M^*\otimes_A M} \circ \mathsf{db}(1_A) \;=\; (\mathsf{db} \otimes_A \mathcal{H}) \circ \varrho_A(1_A),$$

and this finishes the proof. □

The following lemma will be used in the sequel and the $\Bbbk$-algebras involved in it are not necessary commutative.

Lemma 3.10. *Let $(A, \mathfrak{H})$ and $(A', \mathfrak{H}')$ be two corings. Assume that there are two bimodules ${}_BM_A$ and ${}_{B'}M'_{A'}$ such that (M, ϱ_M) and $(M,' \varrho_{M'})$ are, respectively, $(B, \mathfrak{H})$-bicomodule and $(B', \mathfrak{H}')$-bicomodule (here B and B' are considered as B-coring and B'-coring in a trivial way), and that M_A, $M'_{A'}$ are finitely generated and projective modules with dual bases, respectively, $\{m_i, m_i^*\}$ and $\{n_j, n_j^*\}$.*

(1) If the associated canonical map:

$$\mathsf{can}_M : M^* \otimes_B M \to \mathfrak{H}, \quad \Big(m^* \otimes_A m \longmapsto m^*(m_{(0)})m_{(1)}\Big) \tag{27}$$

is injective, then $\mathrm{End}^{M^*\otimes_B M}(M) \;=\; \mathrm{End}^{\mathfrak{H}}(M)$, *where $M^*\otimes_B M$ is the standard A-coring* [4], *or the comatrix A-coring* [12].

(2) If both can_M and $\mathsf{can}_{M'}$ are injective, ${}_BM$ and ${}_{B'}M'$ are faithfully flat modules, then

$$\mathrm{End}^{\mathfrak{H}\otimes\mathfrak{H}'}(M \otimes M') \;\cong\; B \otimes B' \;\cong\; \mathrm{End}^{\mathfrak{H}}(M) \otimes \mathrm{End}^{\mathfrak{H}'}(M').$$

Proof. (1) is a routine computation. (2) uses part (1) and the result [12, Theorem 3.10]. □

3.6. **Dualizable comodule whose endomorphism ring is a principal bundle.** This subsection is of independent interest. We give conditions under which the endomorphism ring (of linear maps) of a dualizable right comodule is a left principal bundle.

Let $(A, \mathcal{H})$ be a flat Hopf algebroid and (M, ϱ) a dualizable right $\mathcal{H}$-comodule. Denote by $B := \mathrm{End}^{\mathcal{H}}(M)$ its endomorphism ring of $\mathcal{H}$-colinear maps, and consider the endomorphism ring of A-linear maps $\mathrm{End}_A(M)$ as right $\mathcal{H}$-comodule via the isomorphism $M^* \otimes_A M \cong \mathrm{End}_A(M)$ together with the following obvious algebra maps

$$\alpha : A \longrightarrow \mathrm{End}_A(M), \quad \beta : B \hookrightarrow \mathrm{End}_A(M).$$

The proof of the following is left to the reader.

Proposition 3.11. *Assume that ${}_AM$, ${}_BM$ are faithfully flat modules and that the canonical map can_M of equation* (27) *is bijective (e.g. when (M, ϱ) is a small generator). Then the triple $(\mathrm{End}_A(M), \alpha, \beta)$ is right principal $(B, \mathcal{H})$-bundle (where (B, B) is considered as a trivial Hopf algebroid).*

4. Geometrically transitive Hopf algebroid: Definition, basic properties and the result

In this section we recall the definition of geometrically transitive Hopf algebroids and prove some of theirs basic properties. Most of the results presented here are in fact consequences of those stated in [4].

4.1. **Definition and basic properties.** For sake of completeness we include the proof of the subsequent.

Proposition 4.1 ([4, Proposition 6.2, page 5845]). *Let $(A, \mathcal{H})$ be a flat Hopf algebroid. Assume that $(A, \mathcal{H})$ satisfies the following condition*

(GT1) $\mathcal{H}$ is projective as an $(A \otimes A)$-module.

Then, we have

(GT11) Every $\mathcal{H}$-comodule is projective as an A-module;
(GT12) $\mathsf{comod}_{\mathcal{H}}$ is an abelian category and the functor $\mathscr{U}_{\mathcal{H}} : \mathsf{comod}_{\mathcal{H}} \to \mathsf{Mod}_A$ is faithful and exact;
(GT13) Every object in $\mathsf{Comod}_{\mathcal{H}}$ is a filtrated limit of subobjects in $\mathsf{comod}_{\mathcal{H}}$.

Proof. $(GT11)$. Let M be a right $\mathcal{H}$-comodule. Then, as a right A-module M is a direct summand of $M \otimes_A \mathcal{H}$. Since ${}_A\mathcal{H}_A$ is a direct summand of a free $(A \otimes A)$-module, M is a direct summand of the right A-module $M \otimes_A (A \otimes A) \cong M \otimes A$. Thus M_A is projective. The same proof works for left $\mathcal{H}$-comodules.

$(GT12)$. The category $\mathsf{comod}_{\mathcal{H}}$ is additive with finite product and cokernels. Let us check that $\mathsf{comod}_{\mathcal{H}}$ do have kernels. So, assume a morphism $f : N \to M$ in $\mathsf{comod}_{\mathcal{H}}$ is given. Then the kernel $\mathrm{Ker}(f)$ is a right $\mathcal{H}$-comodule, since we already know that $\mathsf{Comod}_{\mathcal{H}}$ is a Grothendieck category. Thus we need to check that the underlying module of this kernel is a finitely generated A-module. However, this follows from the fact that $f^k : \mathrm{Ker}(f) \to N$ splits in A-modules, as we know, by the isomorphism of right $\mathcal{H}$-comodules $N/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ and condition $(GT11)$, that this quotient is projective as an A-modules. The last claim in $(GT12)$ is now clear.

$(GT13)$. Following [6, §20.1, §20.2], since ${}_{\mathsf{s}}\mathcal{H}$ is by condition $(GT11)$ a projective module, we have that the category of rational left ${}^*\mathcal{H}$-modules is isomorphic to the category of right $\mathcal{H}$-comodules, where ${}^*\mathcal{H} = \mathrm{Hom}_A({}_{\mathsf{s}}\mathcal{H}, A)$ is the left convolution A-algebra of $\mathcal{H}$. Since any submodule of a rational module is also rational, every rational module is then a filtrated limit of finitely generated submodules. Therefore, any right $\mathcal{H}$-comodule is a filtrated limit of subcomodules in $\mathsf{comod}_{\mathcal{H}}$, as any finitely generated rational module is finitely generated as an A-module. □

Recall that a (locally small) $\Bbbk$-linear category C is said to be *locally of finite type*, if any object in C is of finite length and each of the $\Bbbk$-vector spaces of morphisms $C(c, c')$ is a finite dimensional space.

Definition 4.2. [Bruguières] Let $(A, \mathcal{H})$ be a flat Hopf algebroid. We say that $(A, \mathcal{H})$ is a *geometrically transitive Hopf algebroid* (GT for short) if it satisfies the following conditions:

$(GT1)$ $\mathcal{H}$ is projective as an $(A \otimes A)$-module.
$(GT2)$ The category $\mathsf{comod}_{\mathcal{H}}$ is locally of finite type.
$(GT3)$ $\mathrm{End}^{\mathcal{H}}(A) \cong \Bbbk$.

Therein $\mathrm{End}^{\mathcal{H}}(A)$ denotes the endomorphisms ring of the right $\mathcal{H}$-comodule (A, t) which is identified with the coinvariant subring $A^{coinv_{\mathcal{H}}} = \{a \in A |\ \mathsf{t}(a) = \mathsf{s}(a)\}$.

The subsequent lemma gives others consequences of the properties stated in Definition 4.2, which will be used later on. The proofs are in fact implicitly contained in [4]. For sake of completeness, we give here a slightly different elementary proof.

Lemma 4.3. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid.*

(a) If $(A, \mathcal{H})$ satisfies $(GT11)$ and $(GT3)$, then A is a simple (right) $\mathcal{H}$-comodule.
(b) If $(A, \mathcal{H})$ satisfies $(GT1)$ and $(GT3)$, then every (right) $\mathcal{H}$-comodule is faithfully flat as an A-module.

Proof. (a). First let us check that, under the assumption $(G11)$, any subcomodule of (A, t) is a direct summand in $\mathsf{Comod}_{\mathcal{H}}$. So let (I, ϱ_I) be an $\mathcal{H}$-subcomodule of (A, t). Then A/I is an $\mathcal{H}$-comodule which is finitely generated and projective as an A-module, by assumption $(GT11)$. Therefore, I is a direct summand of A as an A-submodule. Denotes by $\pi : A \to I$ the canonical projection of A-modules, and let $e^2 = e$ be an idempotent element in A such that $I = eA$ and $\pi(a) = ea$, for every $a \in A$.

Next we show that π is a morphism of right $\mathcal{H}$-comodules, which proves that (I, ϱ_I) is a direct summand of (A, t). To this end, it suffices to check that $\mathsf{s}(e) = \mathsf{t}(e)$, since we know that $\mathrm{End}^{\mathcal{H}}(A) = A^{coinv_{\mathcal{H}}} = \{a \in A |\ \mathsf{t}(a) = \mathsf{s}(a)\}$. Clearly the coaction of I is entirely defined by the image of e, and we can write $\varrho_I(e) = e \otimes_A u$, for some element $u \in \mathcal{H}$, which satisfies the following equalities

$$\mathsf{s}(e)\, u \;=\; \mathsf{t}(e) 1_{\mathcal{H}}, \quad e \otimes_A u \otimes_A u \;=\; e \otimes_A u_{(1)} \otimes_A u_{(2)} \in I \otimes_A \mathcal{H} \otimes_A \mathcal{H}, \tag{28}$$

where the first equation comes from the fact that the inclusion $I \hookrightarrow A$ is a morphism of $\mathcal{H}$-comodules.

On the other hand, we know by Lemma 3.9 that I is a dualizable right $\mathcal{H}$-comodule. Its dual comodule have, up to canonical isomorphism, for the underlying A-module $I^* = eA$ with coaction $\varrho_{I^*} : eA \to eA \otimes_A \mathcal{H}$, sending $ea \mapsto e \otimes_A \mathsf{t}(ea) \mathscr{S}(u)$ given by equation (24). The evaluation map $\mathsf{ev} : I^* \otimes_A I \to A$, $ea \otimes_A ea' \mapsto eaa'$ of equation 23, is then a morphism of right $\mathcal{H}$-comodules. Therefore, we have the following equality

$$1 \otimes_A \mathsf{t}(e) \;=\; e \otimes_A \mathsf{t}(e) \mathscr{S}(u) u \;\in\; A \otimes_A {}_{\mathsf{s}}\mathcal{H} \;\cong\; {}_{\mathsf{s}}\mathcal{H}. \tag{29}$$

Combining the first equality of equation (28) and equation 29, we get $\mathsf{t}(e)\mathscr{S}(u) = \mathsf{t}(e)$. Hence $\mathsf{s}(e)u = \mathsf{s}(e)$, and so $\mathsf{s}(e) = \mathsf{t}(e)$, by (28).

We have then show that any $\mathcal{H}$-subcomodule of the $\mathcal{H}$-comodule A is a direct summand, since by $(GT3)$ the endomorphism ring is a field $\mathrm{End}^{\mathcal{H}}(A) \cong \Bbbk$, we conclude that A is a simple $\mathcal{H}$-comodule.

(b). By Proposition 4.1, we know that $(A, \mathcal{H})$ satisfies conditions $(GT11)$-$(GT13)$. Let us first show that any comodule in $\mathsf{comod}_{\mathcal{H}}$ is faithfully flat as an A-module. By condition $(GT11)$, we know that any comodule in this subcategory is finitely generated and projective as A-module, so it is flat as an A-module. Moreover, we know from Lemma 3.9 that the subcategory $\mathsf{comod}_{\mathcal{H}}$ consists exactly of dualizable right $\mathcal{H}$-comodules. Let us then pick a dualizable comodule $M \in \mathsf{comod}_{\mathcal{H}}$, and assume that $M \otimes_A X = 0$ for some A-module X. This in particular implies that $\mathsf{ev}_M \otimes_A X = 0$, from which we get that $A \otimes_A X \cong X = 0$, as ev_M is surjective, since we already know by item (a) that A is a simple comodule. This shows that every object in $\mathsf{comod}_{\mathcal{H}}$ is faithfully flat as an A-module.

For an arbitrary comodule, we know by condition $(GT13)$ stated in Proposition 4.1, that any right $\mathcal{H}$-comodule is a filtrated limit of subcomodules in $\mathsf{comod}_{\mathcal{H}}$. Therefore, any right $\mathcal{H}$-comodule is a flat A-module. Given now a right $\mathcal{H}$-comodule M and assume that $M \otimes_A X = 0$, for some A-module X. We have that $M = \varinjlim(M_i)$ where $\{\tau_{ij} : M_i \hookrightarrow M_j\}_{i \leq j \in \Lambda}$ is a filtrated system in $\mathsf{comod}_{\mathcal{H}}$ with structural morphisms τ_{ij} which are split morphisms of A-modules. This limit is also a filtrated limit of A-modules, and so the equality $\varinjlim(M_i \otimes_A X) \cong M \otimes_A X = 0$ implies that there exists some $j \in \Lambda$, such that $M_j \otimes_A X = 0$. Hence $X = 0$, since M_j is a faithfully flat A-module by the previous argumentation. ◻

Lemma 4.4. *Let $(A, \mathcal{H})$ be a flat Hopf algebroids which satisfies conditions $(GT11)$ and $(GT3)$. Then the $\Bbbk$-algebra map $\eta : A \otimes A \to \mathcal{H}$ is injective. In particular, if $(A, \mathcal{H})$ is geometrically transitive, then η is injective.*

Proof. We know from Lemma 4.3(a) that A is a simple $\mathcal{H}$-comodule. Therefore, by [5, Theorem 3.1], the following map

$$\mathrm{Hom}^{\mathcal{H}}(A, \mathcal{H}) \otimes A \longrightarrow \mathcal{H}, \qquad \Big(f \otimes_{\Bbbk} a \longmapsto f(a)\Big)$$

is a monomorphism, which is, up to the isomorphism $\mathrm{Hom}^{\mathcal{H}}(A, \mathcal{H}) \cong A$ derived from the adjunction between the forgetful functor $\mathscr{U}_{\mathcal{H}}$ and the functor $- \otimes_A \mathcal{H}$, is exactly the map η. Hence η is injective. The particular case is immediately obtained form Definition 4.2 and Proposition 4.1 ◻

We finish this section by characterizing dualizable objects over GT Hopf algebroids.

Proposition 4.5. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid. Assume that $(A, \mathcal{H})$ satisfies the following condition:*

$(GT11)'$ *Every finitely generated right $\mathcal{H}$-comodule is projective.*

Then the full subcategory of $\mathsf{Comod}_{\mathcal{H}}$ of dualizable objects coincides with $\mathsf{comod}_{\mathcal{H}}$. In particular, if $(A, \mathcal{H})$ is geometrically transitive, then the category $\mathsf{comod}_{\mathcal{H}}$ consists of all dualizable right $\mathcal{H}$-comodules.

Proof. By Lemma 3.9, every dualizable right $\mathcal{H}$-comodule is finitely generated and projective as an A-module. This gives the direct inclusion. Conversely, any object in $\mathsf{comod}_{\mathcal{H}}$ is, by condition $(GT11)'$ and Lemma 3.9, a dualizable right $\mathcal{H}$-comodule, form which we obtain the other inclusion. The particular case of GT Hopf algebroids follows directly from Proposition 4.1. ◻

4.2. **Characterization by means of weak equivalences.** This subsection contains our main result. We give several new characterizations of geometrically transitive flat Hopf algebroids. The most striking one is the characterization of these Hopf algebroids by means of weak equivalences, which can be seen as the geometric counterpart of the characterization of transitive abstract groupoids, as we have shown in subsection 2.5, and precisely in Proposition 2.15.

Theorem 4.6. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid over a field $\Bbbk$ such that $A \neq 0$ and $A(\Bbbk) \neq \emptyset$, denote by $\mathscr{H}$ its associated presheaf of groupoids. Then the following are equivalent:*

(i) *$\eta : A \otimes A \to \mathcal{H}$ is a faithfully flat extension;*
(ii) *Any two objects of $\mathscr{H}$ are locally isomorphic, see Definition 3.5;*
(iii) *For any extension $\phi : A \to B$, the extension $\alpha : A \to \mathcal{H}_{\mathsf{t}} \otimes_{A\, \phi} B$, $a \mapsto \mathsf{s}(a) \otimes_A 1_B$ is faithfully flat;*
(iv) *$(A, \mathcal{H})$ is geometrically transitive, see Definition 4.2.*

By [11, Proposition 4.1], condition (iii) in Theorem 4.6 is also equivalent to the following ones:

(v) *For any extension $\phi : A \to B$, the associated canonical morphism of $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{H}_{\phi})$ is a weak equivalence*;

(vi) The trivial principal left $(\mathcal{H}, \mathcal{H}_\phi)$-bundle $\mathcal{H} \otimes_A B$ is a principal bi-bundle.

Given a GT Hopf algebroid $(A, \mathcal{H})$ and an extension $\phi : A \to B$. Since $\mathcal{H}_\phi$ is a flat Hopf algebroid, the forgetful functor $\mathsf{Comod}_{\mathcal{H}_\phi} \to \mathsf{Mod}_B$ is exact. Therefore, condition (v) implies that B is *Landweber exact* over A, in the sense that the functor $\mathscr{U}_{\mathcal{H}}(-) \otimes_A B : \mathsf{Comod}_{\mathcal{H}} \to \mathsf{Mod}_B$ is exact, see [16, Definition 2.1].

Example 4.7. The following Hopf algebroids $(A, A \otimes A)$ and $(A, (A \otimes A)[X, X^{-1}])$ described, respectively, in Examples 3.1 and 3.3, are clearly geometrically transitive. This is also the case of $(A, A \otimes B \otimes A)$ for any Hopf algebra B. On the other hand, if A is a right B-comodule algebra whose canonical map $A \otimes A \to A \otimes B$ is a faithfully flat extension, then the split Hopf algebroid $(A, A \otimes B)$ is obviously geometrically transitive.

A More elaborated example of GT Hopf algebroid, by using principal bundles over Hopf algebras (i.e. Hopf Galois extensions), is given in Proposition 5.11

The proof of $(i) \Rightarrow (ii)$. Let C be an algebra and x, y two objects in $\mathcal{A}(C)$. Denote by $x \otimes y : A \otimes A \to C$ the associated algebra map and consider the obvious algebra map $p : C \to C' := \mathcal{H} \otimes_{A \otimes A} C$. By assumption it is clear that p is a faithfully flat extension. Set the algebra map $g : \mathcal{H} \to C'$ which sends $u \mapsto u \otimes_{A \otimes A} 1_C$. We then have that $p \circ x = g \circ \mathsf{s}$ and $p \circ y = g \circ \mathsf{t}$, which shows that x and y are locally isomorphic.

The proof of $(ii) \Rightarrow (iii)$. We claim that under this hypothesis the underlying A-module of any (left or right) $\mathcal{H}$-comodule is faithfully flat. In particular, this implies that the comodule $\mathcal{H} \otimes_A B$, with coaction $\Delta \otimes_A B$, is faithfully flat for every A-algebra B, and this gives us condition (iii). Since there is an isomorphism of categories between right $\mathcal{H}$-comodules and left $\mathcal{H}$-comodules, which permutes with forgetful functors, it suffices then to show the above claim for right $\mathcal{H}$-comodules.

So let us fix a right $\mathcal{H}$-comodule M and take two objects in different fibres groupoids $x \in \mathcal{A}(T)$ and $y \in \mathcal{A}(S)$, where T, S are arbitrary algebras. We claim that $M \otimes_A {}_xT$ is faithfully flat T-module if and only if $M \otimes_A {}_yS$ is faithfully flat S-module. Clearly our first claim follows from this one since we know that $\mathcal{A}(\Bbbk) \neq \emptyset$ and over a field any module is faithfully flat.

Let us then check this second claim; we first assume that $R = T = S$. In this case, we know that any pair of objects $x, y \in \mathcal{A}(R)$ are locally isomorphic, and thus there exists a faithfully flat extension $p : R \to R'$ and $g \in \mathcal{H}(R)$ such that $\widetilde{x} := p \circ x = g \circ \mathsf{s}$ and $\widetilde{y} := p \circ y = g \circ \mathsf{t}$. On the other hand, the map

$$M \otimes_A {}_{\widetilde{x}}R' \longrightarrow M \otimes_A {}_{\widetilde{y}}R', \quad \Big(m \otimes_A r' \longmapsto m_{(0)} \otimes_A g^{-1}(m_{(1)}) r' \Big)$$

is clearly an isomorphism of R'-modules. Therefore, $M \otimes_A {}_{\widetilde{x}}R'$ is a faithfully flat R'-module if and only if $M \otimes_A {}_{\widetilde{y}}R'$ it is. However, we know that $M \otimes_A {}_{\widetilde{x}}R' \cong (M \otimes_A {}_xR) \otimes_R {}_pR'$ is faithfully flat R'-module if and only if $M \otimes_A {}_xR$ is faithfully flat R-module, as p is a faithfully flat extension. The same then holds true interchanging x by y. Therefore, $M \otimes_A {}_xR$ is faithfully flat R-module if and only if $M \otimes_A {}_yR$ so is.

For the general case, that is, when $T \neq S$ with $x \in \mathcal{A}(T)$ and $y \in \mathcal{A}(S)$, we take $R := T \otimes S$ and consider the canonical faithfully flat extensions $T \to R \leftarrow S$. This leads to the following two objects $\overline{x} : A \to T \to R$ and $\overline{y} : A \to S \to R$. Since $M \otimes_A {}_xT$ (resp. $M \otimes_A {}_yS$) is faithfully flat T-module (resp. S-module) if and only if $M \otimes_A {}_{\overline{x}}R$ (resp. $M \otimes_A {}_{\overline{y}}R$) is faithfully flat R-module, we have, by the proof of the previous case, that $M \otimes_A {}_xT$ is faithfully flat T-module if and only if $M \otimes_A {}_yS$ is faithfully flat S-module, and this finishes the proof of this implication.

The proof of $(iii) \Rightarrow (iv)$. Take an object x in $\mathcal{A}(\Bbbk)$ and denote by $\Bbbk_x$ the base field endowed with its A-algebra structure via the algebra map $x : A \to \Bbbk$. By assumption $A \to \mathcal{H} \otimes_A \Bbbk_x$ is a faithfully flat extension. Therofore, by [11, Proposition 4.1], we know that the associated base change morphism $\mathsf{x} : (A, \mathcal{H}) \to (\Bbbk_x, \mathcal{H}_x)$, where $(\Bbbk_x, \mathcal{H}_x)$ is the Hopf $\Bbbk$-algebra $\mathcal{H}_x = \Bbbk_x \otimes_A \mathcal{H} \otimes_A \Bbbk_x$, is actually a weak equivalence. This means that the induced functor $\mathsf{x}_* := \mathscr{U}_{\mathcal{H}}(-) \otimes_A \Bbbk_x : \mathsf{Comod}_{\mathcal{H}} \to \mathsf{Comod}_{\mathcal{H}_x}$ is a symmetric monoidal equivalence of categories, and thus transforms, up to natural isomorphisms, dualizable $\mathcal{H}$-comodules into dualizable $\mathcal{H}_x$-comodules. Similar property hods true for its inverse functor. In particular, taking an object $M \in \mathsf{comod}_{\mathcal{H}}$, it is clear that $\mathsf{x}_*(M) = M \otimes_A \Bbbk_x$ is finite dimensional $\Bbbk$-vector space and so a dualizable right $\mathcal{H}_x$-comodue, see for instance Lemma 3.9. Therefore, M should be a dualizable right $\mathcal{H}$-comodule. The converse is obvious and then the full subcategory $\mathsf{comod}_{\mathcal{H}}$ coincides with the full subcategory of dualizable right $\mathcal{H}$-comodules, form which we have that $\mathsf{comod}_{\mathcal{H}}$ and $\mathsf{comod}_{\mathcal{H}_x}$ are equivalent $\Bbbk$-linear categories. Hence $\mathsf{comod}_{\mathcal{H}}$ is locally of finite type, and the endomorphism ring $\mathrm{End}^{\mathcal{H}}(A) \cong \Bbbk$. This shows simultaneously conditions $(GT2)$ and $(GT3)$.

To check condition $(GT1)$ we use the morphism between the tensor product Hopf algebroids, that is, $\mathsf{x}^o \otimes \mathsf{x} : (A \otimes A, \mathcal{H}^o \otimes \mathcal{H}) \to (\Bbbk_x \otimes \Bbbk_x \cong \Bbbk, \mathcal{H}_x \otimes \mathcal{H}_x)$. As we have seen in subsection 3.4, this is also a

weak equivalence. Thus the category of right $(\mathcal{H}^o \otimes \mathcal{H})$-comodules is equivalent, as a symmetric monoidal category, to the category of right comodules over the Hopf $\Bbbk$-algebra $\mathcal{H}_x \otimes \mathcal{H}_x$, which as in the case of x also implies that $\mathsf{comod}_{\mathcal{H}^o\otimes\mathcal{H}}$ and $\mathsf{comod}_{\mathcal{H}_x\otimes\mathcal{H}_x}$ are equivalent. Therefore, from one hand, we have by the same reasoning as above that any comodule in $\mathsf{comod}_{\mathcal{H}^o\otimes\mathcal{H}}$ is projective as an $(A \otimes A)$-module since it is a dualizable comodule. On the other hand, we have that every right $(\mathcal{H}^o \otimes \mathcal{H})$-comodule is a filtrated inductive limit of objects in $\mathsf{comod}_{\mathcal{H}^o\otimes\mathcal{H}}$ since right $\mathcal{H}_x \otimes \mathcal{H}_x$-comodules satisfies the same property with respect to finite-dimensional right comodules $\mathsf{comod}_{\mathcal{H}_x\otimes\mathcal{H}_x}$. Now, by apply [4, Proposition 5.1(ii)] to the $(A\otimes A)$-coring $\mathcal{H}^o \otimes \mathcal{H}$, we then conclude that every right $(\mathcal{H}^o \otimes \mathcal{H})$-comodule is projective as an $(A\otimes A)$-module. Thus, $\mathcal{H}$ is projective as an $A \otimes A$-module, which shows condition $(GT1)$.

The proof of $(iv) \Rightarrow (i)$. Set $B := A\otimes A$ and $\mathcal{K} := \mathcal{H}^o \otimes \mathcal{H}$. We know that $(B, \mathcal{K})$ is a flat Hopf algebroid. Since $\mathcal{H}$ is projective as $(A \otimes A)$-module, we have that $\mathcal{K}$ is projective as $(B \otimes B)$-module. Now, since the map η is injective by Lemma 4.4, we can apply Lemma 3.10 by taking $M = A$ as right $\mathcal{H}$-comodule and $M' = A$ as right $\mathcal{H}^o$-comodule, to obtain the following chain of isomorphism

$$\mathrm{End}^{\mathcal{K}}(B) \;=\; \mathrm{End}^{\mathcal{H}^o\otimes\mathcal{H}}(A \otimes A) \;\cong\; \mathrm{End}^{\mathcal{H}^o}(A) \otimes \mathrm{End}^{\mathcal{H}}(A) \;\cong\; \Bbbk \otimes \Bbbk \;\cong\; \Bbbk.$$

This means that the Hopf algebroid $(B, \mathcal{K})$ satisfies the conditions of Lemma 4.3(b). Therefore, any right $\mathcal{K}$-comodule is faithfully flat as a B-module, henceforth, $\mathcal{H}$ is a faithfully flat $A\otimes A$-module. This finishes the proof of Theorem 4.6.

5. More properties of geometrically transitive Hopf algebroids

In this section we give more properties of GT Hopf algebroids. First we set up an analogous property of transitive groupoids with respect to the conjugacy of their isotropy groups. To this end we introduce here perhaps a known notion of *isotropy Hopf algebra*. This is the affine group scheme which represents the presheaf of groups defined by the isotropy group at each fibre. Next we show that any two isotropy Hopf algebras are weakly equivalent. The notion of conjugacy between two isotropy Hopf algebras, is not at all obvious, and the 2-category of flat Hopf algebroids is employed in order to make it clearer. In this direction we show that two isotropy Hopf algebras are conjugated if and only if the characters groupoid is transitive. Lastly, we give an elementary proof of the fact that any dualizable comodule is locally free of constant rank, which in some sense bear out the same property enjoyed by finite dimensional $\Bbbk$-representations of a given transitive groupoid.

5.1. **The isotropy Hopf algebras are weakly equivalent.** Let $(A, \mathcal{H})$ be a flat Hopf algebroid and $\mathscr{H}$ its associated presheaf of groupoids. Assume as before that the base ring satisfies $A \neq 0$ and $A(\Bbbk) \neq \emptyset$, and consider $\mathscr{H}(\Bbbk)$ the characters groupoid of $(A, \mathcal{H})$, see Definition 3.4. To repeat, for each object $x \in A(\Bbbk)$, we denote by $\Bbbk_x$ the A-algebra $\Bbbk$ via the extension x, and consider the associated Hopf $\Bbbk$-algebra of a base ring extension (given by the $\Bbbk$-algebra map $x : A \to \Bbbk_x$), that is, $\mathcal{H}_x := \Bbbk_x \otimes_A \mathcal{H} \otimes_A \Bbbk_x$.

Definition 5.1. Given an object $x \in A(\Bbbk)$. The Hopf algebra $(\Bbbk_x, \mathcal{H}_x)$ is called *the isotropy Hopf algebra of* $(A, \mathcal{H})$ *at the point* x.

It noteworthy to mention, that the associated affine $\Bbbk$-group of $(\Bbbk_x, \mathcal{H}_x)$ coincides with the one called *groupe d'inertie de x relativement à* $\mathscr{H}$ as referred to in [10, III, §2, n$^{\mathrm{o}}$ 2; page 303].

The terminology used in Definition 5.1, is in relation with abstract groupoids justified by the following lemma. Fix an object $x \in A(\Bbbk)$, we denoted by 1_x the unit element of the A-algebra $\Bbbk_x$. Take C to be an algebra with unit map $1_C : \Bbbk \to C$. Composing with x we have then an object $x^*(1_C) = 1_C \circ x \in A(C)$. Let us denote by $\mathscr{G}^x(C) := \mathscr{H}(C)^{x^*(1_C)}$ the isotropy group of the object $x^*(1_C)$ in the groupoid $\mathscr{H}(C)$, see equation (1). This construction is clearly funtorial and so leads to a presheaf of groups $\mathscr{G}^x : C \to \mathscr{G}^x(C)$.

Lemma 5.2. *For any $x \in A(\Bbbk)$, the presheaf of groups $\mathscr{G}^x$ is affine, and up to a natural isomorphism, is represented by the Hopf $\Bbbk$-algebra $\mathcal{H}_x$.*

Proof. Given an element g in the group $\mathscr{G}^x(C)$, that is, an algebra map $g : \mathcal{H} \to C$ such that $g \circ \mathsf{t} = g \circ \mathsf{s} = x^*(1_C)$, we define the following algebra map:

$$\kappa_C(g) : \mathcal{H}_x \longrightarrow C, \quad (k1_x \otimes_A u \otimes_A k'1_x \longmapsto kk'g(u)),$$

which is clearly functorial in C. This leads to a natural transformation $\kappa_- : \mathscr{G}^x(-) \longrightarrow \mathrm{Alg}_{\Bbbk}(\mathcal{H}_x, -)$.

Conversely, to any algebra map $h : \mathcal{H}_x \to C$, one associate the algebra map

$$\nu_C(h) := h \circ \tau_x : \mathcal{H} \longrightarrow \mathcal{H}_x \longrightarrow C,$$

where $\tau_x : \mathcal{H} \to \mathcal{H}_x$ sends $u \mapsto 1_x \otimes_A u \otimes_A 1_x$. This construction is also functorial in C, which defines a natural transformation $\nu_- : \mathrm{Alg}_{\Bbbk}(\mathcal{H}_x, -) \longrightarrow \mathscr{G}^x(-)$. It is not difficult now to check that both natural transformations κ and ν, are mutually inverse. □

To repeat, for abstract groupoids the transitivity property is interpreted by means of conjugation between theirs isotropy groups, which means that any two of these groups are isomorphic. Next we show how this last property is reflected at the level of the isotropy Hopf algebras. The conjugacy of the isotropy Hopf algebras, in relation with the transitivity of the characters groupoid, will be considered in the next subsection.

Proposition 5.3. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid with $A \neq 0$ and $A(\Bbbk) \neq \emptyset$. Assume that $(A, \mathcal{H})$ is geometrically transitive. Then any two isotropy Hopf algebras are weakly equivalent.*

Proof. Take two objects $x, y \in A(\Bbbk)$ and consider as before the following diagram

$$\begin{tikzcd} (\Bbbk_x, \mathcal{H}_x) & & (\Bbbk_y, \mathcal{H}_y) \\ & (A, \mathcal{H}) \arrow[ul, "\mathsf{x}"] \arrow[ur, "\mathsf{y}"'] & \end{tikzcd}$$

of Hopf algebroids. By Theorem 4.6, both x and y are weak equivalences, in particular, the Hopf algebras $(\Bbbk_x, \mathcal{H}_x)$ and $(\Bbbk_y, \mathcal{H}_y)$ are Morita equivalent, in the sense that their categories of comodules are equivalent as symmetric monoidal categories. Therefore, $(\Bbbk_x, \mathcal{H}_x)$ and $(\Bbbk_y, \mathcal{H}_y)$ are weakly equivalent by applying [11, Theorem A]. □

As was shown in [11], any two weakly equivalent flat Hopf algebroids are connected by a two-stage zig-zag of weak equivalences. Thus, in the case of Proposition 5.3, the above diagram can be completed to a square by considering the two-sided translation Hopf algebroid built up by using the principal bibundle connecting $(\Bbbk_x, \mathcal{H}_x)$ and $(\Bbbk_y, \mathcal{H}_y)$. Precisely, we have the two trivial principal bibundles $P_x := \mathcal{H} \otimes_A \Bbbk_x$ and $P_y := \mathcal{H} \otimes_A \Bbbk_y$ which correspond, respectively, to the weak equivalences x and y. Notice that P_x is an $(\mathcal{H}, \mathcal{H}_x)$-bicomodule algebra with algebra maps

$$\alpha_x : A \to P_x, \quad (a \mapsto \mathsf{s}(a) \otimes_A 1) \text{ and } \beta_x : \Bbbk_x \to P_x, \quad (k \mapsto 1_{\mathcal{H}_x} \otimes_A k1_x). \tag{30}$$

Similar notations are applied to the $(\mathcal{H}, \mathcal{H}_y)$-bicomodule algebra P_y. The cotensor product of these two bibundles $P_x{}^{co} \square_{\mathcal{H}} P_y$ is again a principal $(\mathcal{H}_x, \mathcal{H}_y)$-bibundle (recall here that $P_x{}^{co}$ is the opposite bundle of P_x). The algebra maps defining this structure are $\widetilde{\beta_x} : \Bbbk_x \longrightarrow P_x{}^{co} \square_{\mathcal{H}} P_y \longleftarrow \Bbbk_y : \widetilde{\beta_y}$, given by

$$\widetilde{\beta_x}(k) = \beta_x(k) \square_{\mathcal{H}} 1_{P_y}, \qquad \widetilde{\beta_y}(k) = 1_{P_x} \square_{\mathcal{H}} \beta_y(k),$$

where the notation is the obvious one. The associated two-sided translation Hopf algebroid is described as follows. First we observe the following general fact in Hopf algebroids with source equal to the target, i.e. Hopf algebras over commutative rings.

Lemma 5.4. *Let (R, L) and (R', L') be two commutative Hopf algebras, and assume that there is a diagram of Hopf algebroids:*

$$\begin{tikzcd} (R, L) & & (R', L') \\ & (A, \mathcal{H}) \arrow[ul, "\omega"] \arrow[ur, "\omega'"'] & \end{tikzcd}$$

Then the pair $(R \otimes_A \mathcal{H} \otimes_A R', L \otimes_A \mathcal{H} \otimes_A L')$ of algebras, admits a structure of Hopf algebroid with maps:

- *the source and target:*

$$\mathsf{s}(r \otimes_A u \otimes_A r') := r1_L \otimes_A u \otimes_A r'1_{L'}, \quad \mathsf{t}(r \otimes_A u \otimes_A r') := r\omega(\mathscr{S}(u_{(1)})) \otimes_A u_{(0)} \otimes_A \omega'(u_{(1)})r';$$

- *comultiplication and counit:*

$$\Delta(l \otimes_A u \otimes_A l') := (l_{(1)} \otimes_A u \otimes_A l'_{(1)}) \otimes_C (l_{(2)} \otimes_A 1_{\mathcal{H}} \otimes_A l'_{(2)}), \;\; \varepsilon(l \otimes_A u \otimes_A l') := \varepsilon_L(l) \otimes_A u \otimes_A \varepsilon_{L'}(l');$$

• *the antipode:*

$$\mathscr{S}(l \otimes_A u \otimes_A l') \,:=\, \mathcal{S}_L(l\,\omega(u_{(1)})) \otimes_A u_{(2)} \otimes_A \omega'(u_{(3)})\mathcal{S}_{L'}(l').$$

Proof. These are routine computations. □

Now we come back to the situation of Proposition 5.3. Consider the following algebras:

$$P_{x,y} := \Bbbk_x \otimes_A \mathcal{H} \otimes_A \Bbbk_y, \qquad \mathcal{H}_{x,y} := \mathcal{H}_x \otimes_A \mathcal{H} \otimes_A \mathcal{H}_y,$$

with the structure of Hopf algebroid, as in Lemma 5.4. Consider then the following obvious algebra maps

$$\omega_x : \mathcal{H}_x \longrightarrow \mathcal{H}_{x,y},\ \Big(k1_x \otimes_A u \otimes_A k'1_x \longmapsto k1_{\mathcal{H}_x} \otimes_A u \otimes_A k'1_{\mathcal{H}_y}\Big);$$

and

$$\omega_y : \mathcal{H}_y \longrightarrow \mathcal{H}_{x,y},\ \Big(k1_y \otimes_A u \otimes_A k'1_y \longmapsto k1_{\mathcal{H}_x} \otimes_A u \otimes_A k'1_{\mathcal{H}_y}\Big).$$

Proposition 5.5. *Let $(A,\mathcal{H})$ be as in Proposition 5.3, consider $x, y \in A(\Bbbk)$ and their associated isotropy Hopf algebras $(\Bbbk_x, \mathcal{H}_x)$ and $(\Bbbk_y, \mathcal{H}_y)$. Assume that $(A,\mathcal{H})$ is geometrically transitive. Then there is an isomorphism*

$$\Big(P_x^{co} \square_{\mathcal{H}} P_y,\ \mathcal{H}_x \ltimes (P_x^{co} \square_{\mathcal{H}} P_y) \rtimes \mathcal{H}_y\Big) \;\cong\; (P_{x,y}, \mathcal{H}_{x,y})$$

of Hopf algebroids with the following diagram

$$\begin{tikzcd}
 & (P_{x,y}, \mathcal{H}_{x,y}) & \\
(\Bbbk_x, \mathcal{H}_x) \arrow[ur, "\omega_x"] & & (\Bbbk_y, \mathcal{H}_y) \arrow[ul, "\omega_y"'] \\
 & (A, \mathcal{H}) \arrow[ul, "\mathsf{x}"] \arrow[ur, "\mathsf{y}"'] &
\end{tikzcd}$$

of weak equivalences.

Proof. The stated isomorphism follows directly by comparing the structure of the two-sided Hopf algebroid, as given in subsection 3.4, with that of $(P_{x,y}, \mathcal{H}_{x,y})$ given in Lemma 5.4. By Proposition 5.3, we know that x and y are weak equivalences. Therefore, ω_x and ω_y are weak equivalences by applying [11, Proposition 5.3] together with the previous isomorphism of Hopf algebroids. □

Remark 5.6. The diagram stated in Proposition 5.5, is not necessarily strictly commutative; however, it is commutative up to a 2-isomorphism in the 2-category of flat Hopf algebroids. Precisely, one shows by applying [11, Lemma 5.11] that there is a 2-isomorphism $\omega_x \circ \mathsf{x} \;\cong\; \omega_x \circ \mathsf{y}$.

5.2. **The transitivity of the characters groupoid.** Let $(A,\mathcal{H})$ be a flat Hopf algebroid as in the previous subsection and consider its characters groupoid $\mathscr{H}(\Bbbk) = (\mathcal{H}(\Bbbk), A(\Bbbk))$. We have seen in Theorem 4.6 that $(A,\mathcal{H})$ is geometrically transitive if an only if the attached presheaf of groupoids $\mathscr{H}$ is locally transitive, that is, satisfies condition (*ii*) of that theorem. The aim of this subsection is to characterize the transitivity of $\mathscr{H}(\Bbbk)$, as an abstract groupoid, by means of the conjugation between the isotropy Hopf algebras. First we introduce the conjugacy notion.

Definition 5.7. Let x, y be two objects in $\mathscr{H}(\Bbbk)$. We say that *the isotropy Hopf algebras* $(\Bbbk_x, \mathcal{H}_x)$ *and* $(\Bbbk_y, \mathcal{H}_y)$ *are conjugated*, provided there is an isomorphism of Hopf algebras $\mathsf{g} : (\Bbbk_x, \mathcal{H}_x) \to (\Bbbk_y, \mathcal{H}_y)$ such that the following diagram

$$\begin{tikzcd}
(\Bbbk_x, \mathcal{H}_x) \arrow[rr, "\mathsf{g}"] & & (\Bbbk_y, \mathcal{H}_y) \\
 & (A, \mathcal{H}) \arrow[ul, "\mathsf{x}"] \arrow[ur, "\mathsf{y}"'] &
\end{tikzcd}$$

is commutative up to a 2-isomorphism, where Hopf $\Bbbk$-algebras are considered as 0-cells in the 2-category of flat Hopf algebroids described in subsection 3.2.

As in [11, §5.4], this means that there is an algebra map $g : \mathcal{H} \to \Bbbk$ such that

$$g \circ \mathsf{s} = x, \quad g \circ \mathsf{t} = y, \quad \text{and} \quad u_{(1)}{}^{-} \otimes_A u_{(1)}{}^{0} \otimes_A u_{(1)}{}^{+} g(u_{(2)}) \;=\; g(u_{(1)}) \otimes_A u_{(2)} \otimes_A 1_y \;\in \mathcal{H}_y \tag{31}$$

where, by denoting $\mathsf{z} := \mathsf{g} \circ \mathsf{x} : (A, \mathcal{H}) \to (\Bbbk_x, \mathcal{H}_x)$, we have

$$\mathsf{z}_0 = x \quad \text{ and } \quad \mathsf{z}_1(u) = \mathsf{g}(1_x \otimes_A u \otimes_A 1_x) := u^{-} \otimes_A u^{0} \otimes_A u^{+} \text{ (summation understood).}$$

PROPOSITION 5.8. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid with $A \neq 0$ and $A(\Bbbk) \neq \emptyset$. Assume that $(A, \mathcal{H})$ is geometrically transitive. Then the following are equivalent*

(i) the characters groupoid $\mathscr{H}(\Bbbk)$ is transitive;

(ii) for any two object x, y in $\mathscr{H}(\Bbbk)$, the algebras $P_x = \mathcal{H} \otimes_A \Bbbk_x$ and $P_y = \mathcal{H} \otimes_A \Bbbk_y$ are isomorphic as left $\mathcal{H}$-comodules algebras;

(iii) any two isotropy Hopf algebra are conjugated.

Proof. $(i) \Rightarrow (ii)$. Given $x, y \in A(\Bbbk) = \mathscr{H}_0(\Bbbk)$, by assumption there is an algebra map $h : \mathcal{H} \to \Bbbk$ such that $h \circ \mathsf{s} = x$ and $h \circ \mathsf{t} = y$. So we can define the following map

$$F : P_x = \mathcal{H} \otimes_A \Bbbk_x \longrightarrow \mathcal{H} \otimes_A \Bbbk_y = P_y, \qquad \Big(u \otimes_A k1_x \longmapsto u_{(1)} \otimes_A h(\mathscr{S}(u_{(2)}))k1_y \Big).$$

Clearly F is an A-algebra map, and so it is left A-linear. The fact that F is left $\mathcal{H}$-colinear is also clear, and this shows condition (ii), since F is obviously bijective.

$(ii) \Rightarrow (iii)$. Assume for a given $x, y \in A(\Bbbk)$, there is a left $\mathcal{H}$-comodule algebra isomorphism $F : P_x = \mathcal{H} \otimes_A \Bbbk_x \to \mathcal{H} \otimes_A \Bbbk_y = P_y$. For any $u \in \mathcal{H}$, we denote by $F(u \otimes_A 1_x) = u^{-} \otimes_A u^{+}$ (summation understood). Consider the $\Bbbk$-linear map $g : \mathcal{H} \to \Bbbk$ which sends $u \mapsto y(\varepsilon_{\mathcal{H}}(u^{-}))u^{+}$. This is a $\Bbbk$-algebra map since F it is so. For any $a \in A$, we have

$$g(\mathsf{s}(a)) \;=\; y(\varepsilon_{\mathcal{H}}(\mathsf{s}(a)^{-}))\mathsf{s}(a)^{+} \;=\; y(\varepsilon_{\mathcal{H}}(\mathsf{s}(a)))1 \;=\; y(a)$$

and

$$g(\mathsf{t}(a)) \;=\; y(\varepsilon_{\mathcal{H}}(\mathsf{t}(a)^{-}))\mathsf{t}(a)^{+} \;=\; y(\varepsilon_{\mathcal{H}}(1_{\mathcal{H}}))x(a)1 \;=\; x(a),$$

as F is $\Bbbk$-linear. Define the map

$$\mathsf{g} : (\Bbbk_x, \mathcal{H}_x) \longrightarrow (\Bbbk_y, \mathcal{H}_y), \quad \Big((k1_x, 1_x \otimes_A u \otimes_A 1_x) \longmapsto (k1_y, g(\mathscr{S}(u_{(1)}))1_y \otimes_A u_{(2)} \otimes_A g(u_{(3)})1_y) \Big).$$

By using the characterization given in Lemma 5.2, or a direct computation, one can shows that this map is an isomorphism of Hopf algebras. Furthermore, it is easily seen that the pair (g, g) satisfies the equalities of equation (31). Thus, $(\Bbbk_x, \mathcal{H}_x)$ and $(\Bbbk_y, \mathcal{H}_y)$ are conjugated, which means condition (iii).

$(iii) \Rightarrow (i)$. This implication follows immediately from equation (31). □

REMARK 5.9. In case of algebraically closed base field, conditions (1)-(3) of Proposition 5.8 are satisfied at least for a GT Hopf algebroid with finitely generated total algebra. Precisely, assume that $\Bbbk$ is an algebraically closed field and consider $(A, \mathcal{H})$ a GT Hopf algebroid over $\Bbbk$ with $\mathcal{H}$ a finitely generated $\Bbbk$-algebra. Then by condition (i) of Theorem 4.6 and [2, Proposition 9, page 51], the unit map η induces a surjective map $\Omega(\mathcal{H}) \to \Omega(A \otimes A)$ between the sets of maximal ideals, and then a surjective map $\mathcal{H}(\Bbbk) \to A(\Bbbk) \times A(\Bbbk)$ by Zariski's Lemma. Therefore, $\mathscr{H}(\Bbbk)$ is a transitive groupoid, and conditions (1)-(3) of Proposition 5.8 are satisfied.

5.3. **GT Hopf algebroids and principal bundles over Hopf algebras.** Parallel to subsection 2.6 we study here the relationship between GT Hopf algebroids and principal bundles over Hopf algebras (i.e. commutative Hopf Galois extensions [22, §8], [27], or $\Bbbk$-*torsor* as in [14] and [10]). This is a restricted notion of principal bundle, as defined in subsection 3.4, to the case of Hopf algebras.

To be precise, let B be a commutative Hopf algebra over $\Bbbk$, a pair (P, α) consisting of an algebra extension $\alpha : A \to P$ and a right B-comodule algebra P with left A-linear coaction, is said to be a right *principal B-bundle* provided α is faithfully flat and the canonical map $\mathsf{can}_P : P \otimes_A P \to P \otimes H$, $x \otimes_A y \mapsto xy_{(0)} \otimes y_{(1)}$ is bijective. Notice that if we translate this definition to the associated affine $\Bbbk$-schemes, then the outcome characterizes in fact the notion of torsors as it was shown in [10, Corollaire 1.7, page 362], see also [14, Définition 1.4.1, page 117].

Let $(A, \mathcal{H})$ be a Hopf algebroid as in subsection 5.1 and $\mathscr{H}$ its associated presheaf of groupoids. Take an object $x \in A(\Bbbk)$ and consider as before $P_x = \mathcal{H} \otimes_A \Bbbk_x$ the right comodule algebra over the isotropy Hopf algebra $(\Bbbk_x, \mathcal{H}_x)$ with the algebra extension $\alpha_x : A \to P_x$ of equation (30). On the other hand denote by $\mathscr{P}_x$

the presheaf of sets which associated to each algebra C the set $\mathscr{P}_x(C) := \mathsf{t}^{-1}(\{1_C \circ x\})$ where t is the target of the groupoid $\mathscr{H}(C)$.

LEMMA 5.10. *For any $x \in A(\Bbbk)$, the presheaf of sets $\mathscr{P}_x$ is affine, and up to a natural isomorphism, is represented by the algebra P_x. Furthermore, if $(A, \mathcal{H})$ is geometrically transitive, then (P_x, α_x) is a principal right $\mathcal{H}_x$-bundle.*

Proof. The first claim is an immediate verification. The last one is a consequence of Theorem 4.6. □

In contrast with the case of transitive abstract groupoids described in subsection 2.6, the converse in Lemma 5.10 is not obvious. Specifically, it is not automatic to construct a GT Hopf algebroid from a principal bundle over a Hopf algebra. In more details, let (P, α) be a right principal B-bundle over a Hopf algebra B with extension $\alpha : A \to P$, and consider $P \otimes P$ as a right B-comodule algebra via the diagonal coaction and set

$$\mathcal{H} := (P \otimes P)^{coinv_B} = \left\{ u \in P \otimes P |\ \varrho_{P\otimes P}(u) = u \otimes 1_B \right\}$$

its coinvariant subalgebra. The map α induces two maps $\mathsf{s}, \mathsf{t} : A \to \mathcal{H}$ which going to be the source and the target. The counite is induced by the multiplication of P. The comultiplication is derived from that of $(P, P \otimes P)$, however, not in an immediate way, because slightly technical assumptions are needed for this.

Precisely, consider $\mathcal{M} := (P \otimes P) \otimes_A (P \otimes P)$ as a right B-comodule algebra with the coaction

$$\varrho : \mathcal{M} \longrightarrow \mathcal{M} \otimes B, \quad (x \otimes y) \otimes_A (u \otimes v) \longmapsto (x_{(0)} \otimes y_{(0)}) \otimes_A (u_{(0)} \otimes v_{(0)}) \otimes x_{(1)} y_{(1)} u_{(1)} v_{(1)}.$$

This is a well defined coaction since we know that $P^{coinv_B} \cong A$. Clearly we have that $\mathcal{H} \otimes_A \mathcal{H} \subseteq \mathcal{M}^{coinv_B}$, and under the assumption of equality we obtain:

PROPOSITION 5.11. *Let (P, α) be a right principal B-bundle over a Hopf algebra B with extension $\alpha : A \to P$. Denote by $\upsilon : \mathcal{H} := (P \otimes P)^{coin_B} \to P \otimes P$ the canonical injection where $P \otimes P$ is a right B-comodule algebra via the diagonal coaction. Assume that υ is a faithfully flat extension and that $\mathcal{H} \otimes_A \mathcal{H} = \mathcal{M}^{coin_B}$.*

Then $(A, \mathcal{H})$ admits a unique structure of GT Hopf algebroid such that $(\alpha, \upsilon) : (A, \mathcal{H}) \to (P, P \otimes P)$ is a morphism of GT Hopf algebroids.

Proof. First observe that the map $\mathsf{s} : A \to \mathcal{H}$ is a flat extension (and so is t) since α and υ are faithfully flat extension and we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r, "\upsilon"] & P \otimes P \\ 0 \arrow[r] & A \arrow[u, shift left, "\mathsf{t}"'] \arrow[u, shift right, "\mathsf{s}"] \arrow[r, "\alpha"] & P \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

of algebra maps. The fact that $(A, \mathcal{H})$ admits a coassociative comultiplication follows essentially form the second assumption. Indeed, let $\Delta' : P \otimes P \to \mathcal{M}$ be the map which sends $x \otimes y \mapsto (x \otimes 1) \otimes_A (1 \otimes y)$, so it is easily checked that, under the stated assumption, there is a map $\mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$ which completes the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \otimes_A \mathcal{H} \arrow[r] & \mathcal{M} \arrow[r, shift left, "\varrho"] \arrow[r, shift right, "-\otimes 1"'] & \mathcal{M} \otimes B \\ 0 \arrow[r] & \mathcal{H} \arrow[u, dashed, "\Delta"] \arrow[r, "\upsilon"] & P \otimes P \arrow[u, "\Delta'"] & \end{tikzcd}$$

This gives a coassociative comultiplication on the A-bimodule $\mathcal{H}$ using the structure of A-bimodule derived from the above source and the target s, t. To check that Δ is counital one uses the following equalities

$$(p \otimes 1) \otimes_A (1 \otimes q) = \left(p_{(0)} \otimes p_{(1)}^{-} q_{(1)}^{-}\right) \otimes_A \left(p_{(1)}^{+} q_{(1)}^{+} \otimes q_{(0)}\right) \in (P \otimes P) \otimes_A (P \otimes P),$$

together with the properties of the translation map $\delta : B \to P \otimes_A P$, $b \mapsto b^- \otimes_A b^+$ given by the inverse of the canonical map can_P.

With the previous structure maps, $(A, \mathcal{H})$ is now a Hopf algebroid such that the pair of maps $(\alpha, \upsilon) : (A, \mathcal{H}) \to (P, P \otimes P)$ is a morphism of Hopf algebroids, where as was seen before $(P, P \otimes P)$ is GT. Lastly, since $\alpha \otimes \alpha$ is a faithfully flat extension, $\mathsf{s} \otimes \mathsf{t} : A \otimes A \to \mathcal{H}$ is also faithfully flat, and hence $(A, \mathcal{H})$ is by Theorem 4.6 a GT Hopf algebroid as well. □

5.4. **Dualizable comodules over GT Hopf algebroids are locally free of constant rank.** The aim of this subsection is to give an elementary proof of the well know fact sated in [9, page 114] which implicitly asserts that over a GT Hopf algebroid any comodule which has a locally free fibre with rank n, then so are other fibres. An important consequence of this fact is that any dualizable comodule over such a Hopf algebroid is locally free with constant rank. This is an algebraic interpretation of a well known property on representations of transitive groupoid in vector spaces. Namely, if a given representation over such a groupoid have a finite dimensional fibre, then so are all other fibres and all the fibres have the same dimension. We start by the following general lemma which will be needed below.

Lemma 5.12. *Let $\varphi : R \to T$ be a faithfully flat extension of commutative algebras. Then, for any R-module P, the following conditions are equivalent.*

(i) P is locally free R-module of constant rank n;
(ii) $P_{\varphi} := P \otimes_R T$ is locally free T-module of constant rank n.

Proof. By [2, Proposition 12, page 53, and Théorème 1, page 138], we only need to check that P is of a constant rank n if and only if so is P_{φ}. So let us first denote by $\varphi_* : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ the associated continuous map of φ. Denote by $\mathfrak{r}_P^R : \mathrm{Spec}(R) \to \mathbb{Z}$ and $\mathfrak{r}_{P_\varphi}^T : \mathrm{Spec}(T) \to \mathbb{Z}$, the rank functions corresponding, respectively, to P and P_{φ}.

It suffices to check that $\mathfrak{r}_P^R$ is a constant function with value n if and only if $\mathfrak{r}_{P_\varphi}^T$ is a constant function with the same value. Given a prime ideal $\mathfrak{a} \in \mathrm{Spec}(T)$, consider the localising algebras $T_{\mathfrak{a}}$ and $R_{\varphi_*(\mathfrak{a})}$ at the prime ideals $\mathfrak{a}$ and $\varphi_*(\mathfrak{a})$. It is clear that we have an isomorphism of $T_{\mathfrak{a}}$-modules $P \otimes_R T_{\mathfrak{a}} \cong P_{\varphi_*(\mathfrak{a})} \otimes_{R_{\varphi_*(\mathfrak{a})}} T_{\mathfrak{a}}$, where $\varphi_{\mathfrak{a}} : R_{\varphi_*(\mathfrak{a})} \to T_{\mathfrak{a}}$ is the associated localisation map of the extension φ. Therefore, the free modules $P \otimes_R T_{\mathfrak{a}}$ and $P_{\varphi_*(\mathfrak{a})}$ have the same rank. Hence, we have $\mathfrak{r}_P^R(\varphi_*(\mathfrak{a})) = \mathfrak{r}_{P_\varphi}^T(\mathfrak{a})$, for any $\mathfrak{a} \in \mathrm{Spec}(T)$, and so $\mathfrak{r}_P^R \circ \varphi_* = \mathfrak{r}_{P_\varphi}^T$. This shows that if $\mathfrak{r}_P^R$ is a constant function with value n, then so is $\mathfrak{r}_{P_\varphi}^T$. The converse also hods true since we know that φ_* is surjective, because of the faithfully flatness of φ, and this finishes the proof. □

Proposition 5.13. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid with $A \neq 0$ and $A(\mathbb{k}) \neq \emptyset$. Assume that $(A, \mathcal{H})$ is geometrically transitive, and let M be a (right) $\mathcal{H}$-comodule whose underlying A-module is finitely generated and projective. Given two objects $x \in A(S)$ and $y \in A(T)$, then the following are equivalent*

(i) $M_x := M \otimes_A S$ is locally free S-module of constant rank n;
(ii) $M_y := M \otimes_A T$ is locally free T-module of constant rank n.

Proof. Let us first show that the stated conditions are equivalent when $R = S = T$. In this case we know by Theorem 4.6, that the objects $x, y \in A(R)$ are locally isomorphic. Therefore, there exists a faithfully flat extension $p : R \to R'$ such that $M_{\widetilde{x}} = M \otimes_{\widetilde{x}} R'$ is isomorphic as R'-module to $M_{\widetilde{y}} = M \otimes_{\widetilde{y}} R'$, where $\widetilde{x} = p \circ x$ and $\widetilde{y} = p \circ y$. Thus, $M_{\widetilde{x}}$ and $M_{\widetilde{y}}$ they have the same rank function.

On the other hand, by applying Lemma 5.12 to M_x, we get that $M_{\widetilde{x}}$ is locally free R'-module of constant rank n if and only if M_x is locally free R-module of constant rank n. The same result hold true using M_y and $M_{\widetilde{y}}$. Therefore, M_x is locally free R-module of constant rank n if and only if so is M_y.

For the general case $S \neq T$, consider $R := T \otimes_{\mathbb{k}} S$ and set the algebra maps $\overline{x} := \iota_S \circ x$, $\overline{y} := \iota_T \circ y$, where $\iota_S : S \to R \leftarrow T : \iota_T$ are the obvious maps. By the previous case, we know that $M_{\overline{x}}$ is locally free R-module of constant rank n if and only if so is $M_{\overline{y}}$. Now by Lemma 5.12, we have, from one hand, that $M_{\overline{x}}$ is locally free R-module of constant rank n if and only if M_x is locally free S-module of constant rank n, and from the other, we have that $M_{\overline{y}}$ is locally free R-module of constant rank n if and only if M_y is locally free T-module of constant rank n. Therefore, M_x is locally free S-module of constant rank n if and only if M_y is so as T-module. □

As a corollary of Proposition 5.13, we have:

Corollary 5.14. *Let $(A, \mathcal{H})$ be a flat Hopf algebroid with $A \neq 0$ and $A(\mathbb{k}) \neq \emptyset$. Assume that $(A, \mathcal{H})$ is geometrically transitive. Then every dualizable (right) $\mathcal{H}$-comodule is locally free A-module of constant rank. In particular, given a dualizable right $\mathcal{H}$-comodule M and two distinct object $x \neq y \in A(\mathbb{k})$, then M_x and M_y have the same dimension as $\mathbb{k}$-vector spaces.*

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